MATH309-502 (Spring 2013)

Lecture notes for make up class of 02/28/2013

Section 3.6 Row Space, Column Space, Rank of matrix, Rank-Nullity Thm (continued)

Reminder from the last class:

Let A be $m\times n$ matrix

- Column space := span of the columns of A (a subspace of $\mathbb{R}^m$)
- Row space := span of the rows of A (a subspace of $\mathbb{R}^n$)

· rank(A) = dim of the column space $\underbrace{=}_{\text{proved last time}}$ dim of the row space =

= max $\{r :$ there exists nonzero $r\times r$ minor of $A\}$

Nullity of A is the dimension of the null space $N(A)$, i.e. $\dim N(A)$ (recall that $N(A) = \{x\in\mathbb{R}^n : Ax = 0\}$

Rem 1 $\boxed{\text{rank}(A) \le \min(m,n)}$

One method to find rank(A) is to transform A to a row echelon form. Then rank(A) = # of leading element in the row echelon form =

= # of nonzero rows in a row echelon form

Rem 2 Elementary row operations do not change the row space but they in general do change the column space (however they do not change the dimension of the column space)

Rank-Nullity Thm If A is $m\times n$ matrix then

$$\boxed{\underbrace{\text{rank}(A)}_{\substack{\#\text{ of leading}\\ \text{variables}}} + \underbrace{\dim N(A)}_{\substack{\#\text{ of free}\\ \text{variables in r.e.f}}} = n}$$

$\#$ of leading variables + # of free variables in r.e.f $= n$

Column space and consistency of the system $Ax=b$

As we already discussed
$Ax=b$ is consistent $\Leftrightarrow$ b is a linear combination of columns of A $\Leftrightarrow$ b belong to the column space of A

Rem3 $\boxed{Ax=b \text{ is consistent for any } b\in\mathbb{R}^m \Leftrightarrow \text{Column space} = \mathbb{R}^m \Leftrightarrow \operatorname{rank} A = m}$

Nullity and the number of solutions of the system $Ax=b$

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of $Ax=b$ (i.e $A\mathbf{x}_1=b$ and $A\mathbf{x}_2=b$), then $A(\mathbf{x}_1-\mathbf{x}_2)=A\mathbf{x}_1-A\mathbf{x}_2=b-b=0 \Rightarrow$

$\mathbf{x}_1-\mathbf{x}_2\in N(A)$

Via verse, if $\mathbf{x}_p$ is a solution of $Ax=b$ and $\mathbf{x}_h\in N(A)$ then $A(x_p+x_h)=Ax_p+Ax_h=b+0=b \Rightarrow$

$x=x_p+x_h$ is a solution of $Ax=b$

$\Downarrow$

The (general) solution of the system $Ax=b$ (if exists) has the form

x_p+x_h, where x_p is a (particular) solution of $Ax=b$ (if exists) and $x_h\in N_A$ (i.e is a solution of the homogeneous system $Ax=0$)

(compare with what you studied in Differential equations MATH 308)

Therefor if $y_1, \ldots, y_k$ is a basis in $N(A)$, then

$$\underset{\text{of } Ax=b}{\text{gen. sol.}} = \underset{\substack{\text{of } Ax=b \\ \text{(if exists)}}}{\text{part. sol.}} + c_1 y_1 + \ldots + c_k y_k$$

for some constants $c_1, \ldots, c_k$

Conclusion If ~~the~~ system $Ax=b$ is consistent then it has exactly one solution iff $\dim N(A)=0$ and infinitely many solutions iff $\dim N(A)>0$.

Now investigate the number of solutions of $Ax=b$ in various cases:

Case 1 $\underline{\text{rank } A = m}$ ($\Leftrightarrow$ $Ax=b$ is consistent form any b, see Rem. 3, page 2) then we have 2 possibility:

a) $Ax=b$ has exactly one solution for any b.

By Conclusion this happens $\Leftrightarrow$ $\dim N(A)=0$, but by Rank-Nullity theorem $\dim N(A) = n - \text{rank} A = n-m$, therefore

$$\dim N(A)=0 \Leftrightarrow n=m$$

So, if $\text{rank } A = m$ then the system $Ax=b$ has exactly one solution $\Leftrightarrow$ $n=m$

b) $Ax=b$ has infinite many solutions $\overset{\text{Conclusion}}{\Leftrightarrow}$ $\dim N(A)>0$ $\Leftrightarrow$ $n - \text{rank}(A) = n-m>0$, i.e. $m<n$

(Note that if $\text{rank } A = m$ then $n \geq m$, (because $\text{rank}(A) \leq n$))

Case 2 rank $A < m$

a) This means that the column space is not the whole R^m $\Rightarrow$ there exist $b \in \mathbb{R}^m$ which are not in the column space and the system $Ax = b$ is inconsistent for such b $\Leftrightarrow$ no solution.

b) Now assume that b is in column space. Then we have two additional subcases:

(b1) $\text{rank}(A) = n \overset{\text{Rank-Nullity Thm}}{\Longrightarrow} \dim N(A) = n - \text{rank}(A) = 0$

$\Rightarrow Ax = b$ has exactly one solution

(b2) $\text{rank}(A) < n \overset{\text{Rank-Nullity Thm}}{\Longrightarrow} \dim N(A) = n - \text{rank}(A) > 0$

$\Rightarrow Ax = b$ has infinite many solutions

We can summarize our our analysis in the following table

$Ax = b$ has

	no solutions	exactly one solution	infinite many solutions
1. rank $A = m = n$	impossible	always happens	impossible
2. rank $A = m < n$	impossible	impossible	always happens
3. rank $A < m$ & rank $A = n$	possible ($\Leftrightarrow$ b is not in column space)	possible ($\Leftrightarrow$ b is in column space)	impossible
4. rank $A < m$ & rank $A < n$	possible ($\Leftrightarrow$ b is not in column space)	impossible	possible ($\Leftrightarrow$ b is in column space)

Rem rank $A = m = n \iff A$ is nonsingular

Rem rank $A = n \iff$ # of column vectors = dim of column space $\iff$ column vectors form a basis for the column space

Example (problem 8, p. 160)

Let A be an $m\times n$ matrix with $m > n$. Let $b \in \mathbb{R}^m$ and suppose that $N(A) = 0$.

(a) What can you conclude about the column vectors of A: i) are they linearly independent? Yes

$\operatorname{rank}(A) = n - \dim N(A) = n$

ii) do they span $\mathbb{R}^m$? No

$\operatorname{rank}(A) = n < m \Rightarrow$ the column space is not the whole $\mathbb{R}^m$

(b) i) How many solutions will the system $Ax = b$ have if b is not in the column space? Answer: No solutions

ii) How many solutions there will be if b is in the column space? Answer: Exactly one (Because $\dim N(A) = 0$ or see row 3 of the table of the previous page.

The notes are continued in the next page!

How to find the rank, the nullity, the basis of row space, column space and null space?

We will explain this through example:

Example Given the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix}$$

Find

a) rank A

Transform A to row echelon form

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix} \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 + 2R_1} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix} \xrightarrow[R_4 \to R_4 - 3R_2]{R_3 \to R_3 + R_2}$$

Leading elements are marked

$$\to \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{R_4 \to \frac{1}{3}R_4} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# of leading element (= # of non zero rows) $= 3 \Rightarrow \boxed{\operatorname{rank}(A) = 3}$

b) Nullity of A (i.e. $\dim N(A)$)

By the Rank-Nullity Thm

$$\underbrace{\operatorname{rank}(A)}_{3} + \dim N(A) = n = 5 \Rightarrow$$

$$\dim N(A) = 5 - \operatorname{rank}(A) = 5 - 3 = \boxed{2}$$

c) Find a basis of the row space: As a basis one can take all non-zero rows in the row echelon form, i.e.

$$\boxed{w_1 = (1,0,-2,1,0),\ w_2 = (0,1,3,-1,-3),\ w_3 = (0,0,0,1,-1)}$$

d) Find a basis of column space:
Take the columns of the original matrix, corresponding to the leading elements, i.e. columns #1, #2, #4:

$$\boxed{v_1 = \begin{pmatrix}1\\0\\-2\\0\end{pmatrix},\ v_2 = \begin{pmatrix}0\\-1\\-1\\3\end{pmatrix},\ v_4 = \begin{pmatrix}1\\1\\-1\\0\end{pmatrix}}$$

(In contrast to the basis of the row space we take the columns of the original matrix, see Remark 2 on page 1)

e) Find a basis of null space
For this consider the homogeneous system. Since we already have transformed our matrix A into the row echelon form in item a), the row echelon form of the augmented matrix for the homogeneous system is

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow$$

(x_3 and x_5 are free variables)

$$\begin{cases} x_1 - 2x_3 + x_4 = 0 & x_1 = 2x_3 - x_4 = 2x_3 - x_5 \\ x_2 + 3x_3 - x_4 - 3x_5 = 0 & x_2 = -3x_3 + x_4 + 3x_5 = -3x_3 + 4x_5 \\ x_4 - x_5 = 0 \Rightarrow & x_4 = x_5 \end{cases}$$

$$\Rightarrow N(A) = \left\{ \begin{pmatrix} 2x_3 - x_5 \\ -3x_3 + 4x_5 \\ x_3 \\ x_5 \\ x_5 \end{pmatrix} \middle|\ x_3, x_5 \in \mathbb{R} \right\} = \left\{ x_3 \begin{pmatrix}2\\-3\\1\\0\\0\end{pmatrix} + x_5 \begin{pmatrix}-1\\4\\0\\1\\1\end{pmatrix} \middle|\ x_3, x_5 \in \mathbb{R} \right\}$$

$\Rightarrow$ $N(A)$ is spanned by 2 vectors $\begin{pmatrix}2\\-3\\1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\4\\0\\1\\1\end{pmatrix}$

Note that these vectors are linearly independent: 2×2 minors of the matrix $\begin{pmatrix}2 & -1\\-3 & 4\\1 & 0\\0 & 1\\0 & 1\end{pmatrix}$ corresponding to rows #3 and #5 (the rows with the same number as the index of the free variables) is $\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} = 1 \neq 0$), so

$\underbrace{\begin{pmatrix}2\\-3\\1\\0\\0\end{pmatrix}}_{\text{corresponds to } x_3=1,\ x_5=0}$ and $\underbrace{\begin{pmatrix}-1\\4\\0\\1\\1\end{pmatrix}}_{\text{corresponds to } x_3=0,\ x_5=1}$ form a basis in $N(A)$

<u>Remark</u> In general, if $x_{j_1}, x_{j_2}, \dots, x_{j_k}$ are the free variables then the k column vectors of solutions of the homogeneous equation corresponding to tuples $(x_{j_1}, x_{j_2}, \dots, x_{j_k})$ with unique 1 and all other entries equal to zeros form a basis of $N(A)$.

In other words, find the general solution of $Ax=0$ and then substitute 1 for one of the ~~free~~ variables and 0 for all others. Then the vectors obtained in this way form a basis in $N(A)$.

f) Let v_j is the jth column of A.

Represent v_3 and v_5 as a linear combination of v_1, v_2 and v_4 (we can do it because v_1, v_2 and v_4 form a basis of the column space)

One way

i) For V_3 we have to find x_1, x_2 and x_4 such that

$$x_1V_1 + x_2V_2 + x_4V_4 = V_3 \iff x_1V_1 + x_2V_2 - V_3 + x_4V_4 = 0$$

$\Rightarrow$ It corresponds to the solution of homogeneous system $Ax = 0$ with $x_3 = -1$ and $x_5 = 0$ $\Rightarrow$ (see the last formula of page 7): $x_1 = -2$, $x_2 = 3$, $x_4 = 0$ $\Rightarrow$

$$\boxed{V_3 = -2V_1 + 3V_2}$$

ii) For V_5 we have to find x_1, x_2 and x_4 such that

$$x_1V_1 + x_2V_2 + x_4V_4 = V_5 \iff x_1V_1 + x_2V_2 + x_4V_4 - V_5 = 0 \Rightarrow$$

It corresponds to the solution of homogeneous system $Ax = 0$ with $x_3 = 0$, $x_5 = -1$ $\Rightarrow$ (see the last formula of page 7): $x_1 = 1$, $x_2 = -4$, $x_4 = -1$ $\Rightarrow$

$$\boxed{V_5 = V_1 - 4V_2 - V_4}$$

Another way Put A to the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_3} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the first 3 entries of the third column give the coefficients of the linear combination of V_3 w.r.t. V_1, V_2, V_4: $V_3 = -2V_1 + 3V_2$

and the first 3 entries of the fifth column give the coefficients of the linear combination of V_5 w.r.t. V_1, V_2, V_5: $V_5 = V_1 - 4V_2 - V_4$