Examples for my section of M309

Example 1

Let L be a linear transformation mapping R^3 into R^2 defined by

$$L(x) = (-2x_1 + x_2 - 3x_3)b_1 + (x_2 + 4x_3)b_2$$

for each $x \in R^3$, where

$$b_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \text{ and } b_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Find the matrix A representing L with respect to the ordered bases $\{e_1, e_2, e_3\}$ and $\{b_1, b_2\}$

Solution

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$L(e_1) = -2b_1 + 0 \cdot b_2$$
$$L(e_2) = b_1 + b_2$$
$$L(e_3) = -3b_1 + 4b_2$$

The jth column of A is equal to the coordinate vector of $L(e_j)$ w.r.t. the basis $\{b_1, b_2\}$ $\Rightarrow$

$$A = \begin{pmatrix} -2 & 1 & -3 \\ 0 & 1 & 4 \end{pmatrix}$$

<u>Example 2</u> Let $L: P_3 \to P_3$ is given by
(similar to example 5 p. 180)

$$L(p(x)) = p'(x) - 2p(x)$$

a) Is it a linear transformation

b) What is the matrix representation of L in the basis $E = \{1, x, x^2\}$ of P_3 (i.e. in the standard basis of P_3)

Solution of b) $L(1) = (1)' - 2 \cdot 1 = 0 - 2 = -2 \Rightarrow$

$$[L(1)]_E = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$L(x) = (x)' - 2x = 1 - 2x \Rightarrow$$

$$[L(x)]_E = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$L(x^2) = (x^2)' - 2x^2 = 2x - 2x^2 \Rightarrow$$

$[L(x^2)] = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \Rightarrow$ the matrix representing L in the standard basis is $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$

Example 3 (similar to example 6, p. 181)

Let $L: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation

$$L(x) = (x_1 - x_2, -x_1 + 2x_2, 3x_1 - 2x_2)^T$$

Find the matrix representation of L with respect to the ordered bases $u_1 = \begin{pmatrix}1\\2\end{pmatrix}, u_2 = \begin{pmatrix}3\\2\end{pmatrix}$ in $\mathbb{R}^2$

and $b_1 = (1,1,-1)^T, b_2 = (1,1,-2), b_3 = (-1,1,2)$ in $\mathbb{R}^3$

Calculations Let $B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & -2 & 2 \end{pmatrix}$, $\underbrace{A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & -2 \end{pmatrix}}$, $U = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

the matrix of A w.r.t. the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$

Then the required matrix representation of L is equal to

$$B^{-1}AU$$

One of the method of calculation of this matrix is to transform the matrix $(B \mid AU)$ to the reduced row echelon form (then it has exactly the form $(I \mid B^{-1}AU)$)

$$AU = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & 1 \\ -1 & 5 \end{pmatrix}$$

$$(B \mid AU) = \left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ -1 & -2 & 2 & -1 & 5 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & -1 & 1 & -2 & 6 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & -2 & 6 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right) \xrightarrow[R_3 \to \frac{1}{2}R_3]{R_2 \to -R_2} \left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & 2 & -6 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -3 & 7 \\ 0 & 1 & 0 & 4 & -6 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

→ the answer

Example 4 Given $L: \mathbb{R}^2 \to \mathbb{R}^2$

$$L(X) = (-11x_1 + 4x_2,\ 30x_1 + 11x_2)$$

a) find the matrix representing L in the basis $u_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and

Solution: $L(x) = \underbrace{\begin{pmatrix} -11 & -4 \\ 30 & 11 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

The transition matrix from (u_1, u_2) to the standard basis (e_1, e_2) is

$$U = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

The required matrix is equal to $U^{-1}AU$.

$$U^{-1} = \frac{1}{6-5}\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

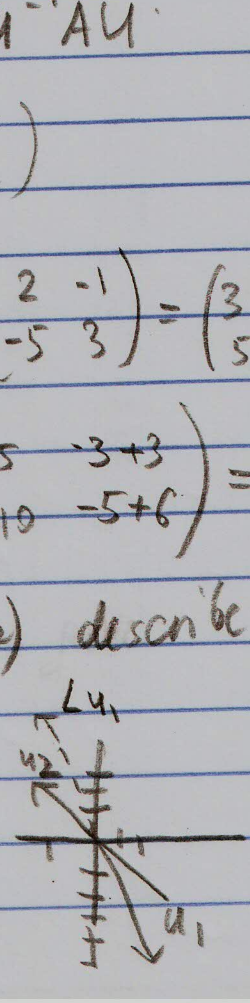
$$U^{-1}AU = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -11 & -4 \\ 30 & 11 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -22+20 & 11-12 \\ 60-55 & -30+33 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} -6+5 & -3+3 \\ -10+10 & -5+6 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) Based on the results of a) describe what transformation L does on $\mathbb{R}^2$?

$L(u_1) = -u_1$, $L(u_2) = u_2$

L fix all points of the line $\mathrm{Span}(u_2)$



To find Lv represent $v = y_1u_1 + y_2u_2$

then $Lv = -y_1u_1 + y_2u_2$