Supplementary lecture notes/ worksheets in MATH309 - Spring 2013
for the classes of Igor Zelenko of February 1 and 4
Definition of determinant using permutations

Print these worksheets and bring them to the classes of February 1 and 4. Try to read them (at least the first four pages) and to answer the questions below before the class of Friday, February 1. You have an option to submit these sheets with your answers to the questions below for the extra credit at the beginning of the Friday class. You may submit only first four pages, there is no questions to answer in the rest. In this case please photocopy these four pages before your submission in order to have them during the class because we will work on them. The submission will weigh totally a half of a homework assignment. A partial credit will be given as well.

The definition of the determinant of $n \times n$ matrix by induction via the expansion along the first row, given in the textbook on page 89, is probably the quickest way to introduce this notion. However, this definition is not convenient for explaining the main properties of the determinant and this is probably the main reason why for example Theorem 2.1.1 (in the same page of the textbook right after this definition) about the expansion along any row and column is neither proved nor even explained there, while the proof of other results (for example the next Theorem 2.1.2) is heavily based on the same unproved Theorem 2.1.1.

Below I describe another (very standard/common/classical) way to define the determinant. Although this way seems to be more complicated at first glance and needs some additional concepts from Combinatorics, it is significantly elucidate the notion of the determinant and its properties. So, I am convinced that the additional efforts needed to understand this definition and the combinatorial notions related to it will well paid off for your final understanding of this topic.

1. Permutations, inversions, and signature of permutation

Definition 1. *An array $(j_1 j_2 \ldots j_n)$ of integers from 1 to n such that each number from 1 to n appears exactly once is called a <u>permutation (of first n integers)</u>. In other words, a permutation is an array when the first n integers are written in some order.*

For example $(3\,2\,1)$ is a permutation but $(1\,2\,1)$ is not, because 3 does not appear there (or because 1 appears twice).

If $n = 2$ there are 2 permutations: $(1\,2)$ and $(2\,1)$.

Question 1. If $n = 3$

- list all permutations:

- how many permutation do you have?

Question 2. For general n how many permutation (of first n integers) are there?

How are permutations related to the determinant? Let us look more carefully in the expansions of the determinant of 2×2 and 3×3 matrices (that you learned in your previous courses starting from High School Algebra 2 and also in Engineering Calculus 3 and Differential Equations):

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{12} - a_{12}a_{21} \tag{1}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \tag{2}$$

In both cases the determinant is a sum of $n!$ terms of the form

$$\pm a_{1j_1} a_{2j_1} \ldots a_{nj_n}$$

($\pm$ means that it is taken either with $+$ or with $-$), where $(j_1 \, j_2 \ldots j_n)$ is a permutation of first n integers.

For example

- the term $a_{11}a_{22}$ in the right hand side of equation (1) corresponds to the permutation $(1\,2)$ and it is taken with $+$,

- the term $a_{12}a_{21}$ in the right hand side of equation (1) corresponds to the permutation $(2\,1)$ and it is taken with $-$,

- the term $a_{13}a_{21}a_{32}$ in the right hand side of equation (2) corresponds to the permutation $(3\,1\,2)$ and it is taken with $+$,

Question 3. To what permutation does the term $a_{13}a_{22}a_{31}$ in the right hand side of equation (2) correspond?

Note that it is taken with $-$ there.

The key point is that the same expansion of the determinant is valid for general n. The big question is: **With what sign, $+$ or $-$, the term** $a_{1j_1} a_{2j_1} \ldots a_{nj_n}$ **should be taken?**

In order to answer this question we need the following

Definition 2. *A pair of numbers (j_k, j_l) in the permutation $(j_1 \, j_2 \, \ldots \, j_n)$ is called an <u>inversion</u> if $k < l$ and $j_k > j_l$. In other words, if a larger number precedes a smaller one in the permutation $(j_1, j_2, \ldots, j_n)$ then these two numbers create an inversion.*

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Definition 3. *The number*

$$\sigma(j_1, j_2, \ldots j_n) = (-1)^{\# \text{ of inversions in } (j_1, j_2, \ldots, j_n)}$$

is called the __signature__ of the permutation $(j_1 \, j_2 \, \ldots \, j_n)$. *In other words*

$$\sigma(j_1, j_2, \ldots j_n) = \begin{cases} 1 & \text{if } \# \text{of inversions in } (j_1, j_2, \ldots, j_n) \text{ is even} \\ -1 & \text{if } \# \text{of inversions in } (j_1, j_2, \ldots, j_n) \text{ is odd} \end{cases}$$

In the first case the permutation is called __even__ and in the second case it is called __odd__.

For example,

- In the permutation $(1\,2)$ there is no inversion, so it is an even permutation and its signature is 1;

- In the permutation $(2\,1)$ there is 1 inversion: $(2,1)$, so it is an odd permutation and $\sigma(2,1) = -1$;

- In the permutation $(3\,1\,2)$ the inversions are $(3,1)$ and $(3,2)$. So, there are 2 inversions, the permutation is even and its signature is equal to 1;

Question 4. Given the permutation $(3\,2\,1)$

- List all its inversions:

- Decide whether it is an even or odd permutation:

- Find its signature:

Question 5. Given the permutation $(3\,4\,2\,1)$

- List all its inversions:

- Decide whether it is an even or odd permutation:

- Find its signature:

Question 6. Given the permutation $(5\,2\,1\,4\,3)$

- List all its inversions:

- Decide whether it is an even or odd permutation:

- Find its signature:

Now we are ready to give the general definition of the determinant:

Definition 4. *Given an $n \times n$ matrix $A = (a_{ij})$ the determinant $det(A)$ is the following number*

$$\det(A) := \sum_{\text{all permutations } (j_1\, j_2\, \dots\, j_n)} \sigma(j_1, \dots, j_n) a_{1j_1} a_{2j_1} \dots a_{nj_n} \tag{3}$$

In other words, the sign of the term $a_{1j_1} a_{2j_1} \dots a_{nj_n}$ in the expansion of the determinant is equal to the signature of the permutation $(j_1\, j_2\, \dots\, j_n)$.

Question 7. Show that Definition 4 matches our previous definition of the determinant

- for $n = 2$

- for $n = 3$

I will explain on the whiteboard why the inductive definition via the expansion along the first row given previously matches Definition 4 (take the notes in your notebook).

Question 8. Let $A = (a_{ij})$ be a 5×5 matrix. Determine with what sign appear the term $a_{15} a_{21} a_{32} a_{44} a_{53}$ in the expansion of the determinant of A: For this you need to answer the following questions:

- To what permutation (of first 5 integers) does this term correspond?

- List all inversions of this permutation:

- What is the signature of this permutation and therefore the answer to the original question?:

The expansion (3) is not convenient for calculation of the determinant: you need to perform $n \cdot n!$ multiplications and $n! - 1$ additions/subtractions in order to do that (figure out what are these numbers already for $n = 5$) but it is very convenient in order to understand the properties of the determinant and in this way to justify other, more efficient ways for calculations of the determinant (based on the row/column operations).

2. Transpositions

Below I will focus on several additional combinatorial aspects related to the notion of signature of a permutation that will help us in justifying Theorem 2.1.1 and the rest of the material of Sections 2.1 and 2.2. I introduce the important notion of a transposition of a permutation and state and prove (for completeness) two important properties of them which are the basis for the theory of determinants. You may skip the proofs but I still recommend to read them, because I tried to make them as transparent and short as possible. I also will try to explain (also graphically) the main ideas of them on the whiteboard.

Definition 5. *An interchanging of any two elements in a permutation (with all other elements remaining in their places) is called a <u>transposition</u> . An interchanging of any two adjacent elements in a permutation (with all other elements remaining in their places) is called an <u>elementary transposition</u>.*

For example, $(2\,1\,3\,4) \mapsto (2\,4\,3\,1)$ is a transposition because we interchange 1 and 4 with 2 and 3 remaining in their places but it is not an elementary transposition, because 1 and 4 are not adjacent in $(2\,1\,3\,4)$. However, you can represent this transposition as a series (a composition) of 3 elementary transpositions:

$$(2\,1\,3\,4) \mapsto (2\,3\,1\,4) \mapsto (2\,3\,4\,1) \mapsto (2\,4\,3\,1) \tag{4}$$

(first we swap 1 and 3, then 1 and 4, i.e. in this way we move 1 to the forth place, and finally we swap 3 and 4).

Proposition 1. *A transposition changes the parity of a permutation, i.e. if we perform a transposition in an even permutation then the resulting permutation becomes odd and if we perform a transposition in an odd permutation then the resulting permutation becomes even.*

Proof.
1. After an elementary transposition the number of inversion increases by 1 or decreases by 1. Indeed, if you swap two adjacent elements such that initially the larger one was on the right, then one new inversion is created (and the rest remains the same) and if you swap two adjacent elements such that initially the larger one was on the left, then one inversion disappears (and the rest remains the same). So, in both cases the parity of the permutation changes.
2. Any transposition is a series (a composition) of an odd number of elementary transpositions (as in (4)): Indeed, if a transposition interchanges j_k and j_l (where $k < l$) then using $l - k$ elementary transpositions you move j_k to the lth place, then j_l became in the $(l-1)$st place and so you need other $l - k - 1$ elementary transpositions to move j_l to the kth place. Thus we used $l - k + l - k - 1 = 2(l-k) - 1$ transpositions in total, which is an odd number. By part 1 of the proof each elementary transposition changes the parity of the permutations. Therefore the odd number of elementary transpositions changes the parity of the permutations. $\square$

Using Definition 4 and Proposition 1 it is almost immediate to show that **interchanging any row in a matrix changes the sign of the determinant**, then also to prove the part of (unproved) Theorem 2.1.1, page 89, regarding the expansion along an arbitrary row and other

properties listed in sections 2.1 and 2.2 (I will formulate and justify all these properties on the whiteboard, take notes in your notebook).

Another useful property of transpositions is given in the following

Proposition 2. *The parity of a permutation $(j_1\,j_2\,\ldots\,j_n)$ is equal to the parity of a number of transposition needed to obtain from this permutation the permutation $(1\,2\,\ldots n)$*

Proof. **1.** First, let us describe one way to transform the permutation $(j_1\,j_2\,\ldots\,j_n)$ to the permutation $(1\,2\,\ldots n)$ by a series of transpositions such that the number of transposition used in it is exactly equal to the number of inversions in $(j_1\,j_2\,\ldots\,j_n)$. For this note that the permutation $(1\,2\,\ldots n)$ is the only permutation without inversions. If $(j_1\,j_2\,\ldots\,j_n) = (1\,2\,\ldots n)$ there is nothing to do. Otherwise, there exist at least one inversion in $(j_1\,j_2\,\ldots\,j_n)$ and therefore there exist at least one inversion in $(j_1\,j_2\,\ldots\,j_n)$ consisting of adjacent elements. Swapping the elements in this inversion we decrease the number of inversions exactly by 1. We may repeat the process a finite number of times until we get 0 inversions (or, equivalently, the permutation $(1\,2\,\ldots n)$) decreasing on each step the number of inversions exactly by 1 using exactly one elementary transposition. Moreover, by the latter the total number of times we do this is equal on one hand to the number of inversions in $(j_1\,j_2\,\ldots\,j_n)$ and on the other hand to the number of transposition used. It shows that indeed we can transform the permutation $(j_1\,j_2\,\ldots\,j_n)$ to the permutation $(1\,2\,\ldots n)$ by the number of transpositions equal to the number of inversions in $(j_1\,j_2\,\ldots\,j_n)$.

2 Note that there are many ways to transform the permutation $(j_1\,j_2\,\ldots\,j_n)$ to the permutation $(1\,2\,\ldots n)$ but the parity of the number of transpositions for all such ways must be the same, otherwise by Proposition 1 $(j_1\,j_2\,\ldots\,j_n)$ will be both even and odd, which is impossible. Therefore from part 1 of the proof it follows that the parity of the number of transpositions in any such way is equal to the parity of the number of inversions in $(j_1\,j_2\,\ldots\,j_n)$, because these parity are equal at least for one such way (described in part 1). On the other hand, the parity of the number of inversions in $(j_1\,j_2\,\ldots\,j_n)$ is equal by Definition 3 to the parity of the permutation $(j_1\,j_2\,\ldots\,j_n)$, which completes the proof of the proposition. $\square$

Using Proposition 2 one can prove almost immediately the part of (unproved) Theorem 2.1.1, page 89, regarding the expansion along an arbitrary column which in turn shows that all properties of the determinant related to the row operations are automatically valid for the corresponding column operations (will be discussed on the whiteboard in more detail).