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where $P(s)$ and $Q(s)$ are polynomials with $\deg P(s) < \deg Q(s)$.

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$$\mathcal{L}^{-1}\left\{\frac{A(s-\alpha)+B\beta}{(s-\alpha)^2+\beta^2}\right\} = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t;$$

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where $1 \leq i \leq m$.

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The calculation of the inverse Laplace transform in this case is more involved. It can be done as a combination of the property of the derivative of Laplace transform and the notion of *convolution* that will be discussed in section 6.6 or using decomposition to linear factors using complex roots as in Enrichment 8.