Inverse Laplace transform of rational functions using Partial Fraction Decomposition

Using the Laplace transform for solving linear non-homogeneous differential equation with constant coefficients and the right-hand side $g(t)$ of the form

$h(t)e^{\alpha t}\cos\beta t$ or $h(t)e^{\alpha t}\sin\beta t$,

where $h(t)$ is a polynomial, one needs on certain step to find the inverse Laplace transform of rational functions $\dfrac{P(s)}{Q(s)}$,

where $P(s)$ and $Q(s)$ are polynomials with $\deg P(s) < \deg Q(s)$.

Inverse Laplace transform of rational functions using Partial Fraction Decomposition

The latter can be done by means of the partial fraction decomposition that you studied in Calculus 2:

One factors the denominator $Q(s)$ as much as possible, i.e. into linear (may be repeated) and quadratic (may be repeated) factors:

each linear factor corresponds to a real root of $Q(s)$ and each quadratic factor corresponds to a pair of complex conjugate roots of $Q(s)$.

Each factor in the decomposition of $Q(s)$ gives a contribution of certain type to the partial fraction decomposition of $\frac{P(s)}{Q(s)}$. Below we list these contributions depending on the type of the factor and identify the inverse Laplace transform of these contributions:

Case 1 A non-repeated linear factor $(s-a)$ of $Q(s)$ (corresponding to the root a of $Q(s)$ of multiplicity 1) gives a contribution of the form $\frac{A}{s-a}$. Then $\mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{at}$;

Case 2 A repeated linear factor $(s-a)^m$ of $Q(s)$ (corresponding to the root a of $Q(s)$ of multiplicity m) gives a contribution which is a sum of terms of the form $\frac{A_i}{(s-a)^i}$, $1 \leq i \leq m$. Then $\mathcal{L}^{-1}\left\{\frac{A_i}{(s-a)^i}\right\} = \frac{A_i}{(i-1)!}t^{i-1}e^{at}$;

Case 3 A non-repeated quadratic factor $(s-\alpha)^2+\beta^2$ of $Q(s)$ (corresponding to the pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity 1) gives a contribution of the form

$$\frac{Cs+D}{(s-\alpha)^2+\beta^2}.$$

It is more convenient here to represent it in the following way:

$$\frac{Cs+D}{(s-\alpha)^2+\beta^2}=\frac{A(s-\alpha)+B\beta}{(s-\alpha)^2+\beta^2}.$$ Then

$$\mathcal{L}^{-1}\left\{\frac{A(s-\alpha)+B\beta}{(s-\alpha)^2+\beta^2}\right\}=Ae^{\alpha t}\cos\beta t+Be^{\alpha t}\sin\beta t;$$

Case 4 A repeated quadratic factor $\left((s-\alpha)^2+\beta^2\right)^m$ of $Q(s)$ (corresponding to the pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity m) gives a contribution which is a sum of terms of the form

$$\frac{C_i s + D_i}{\left((s-\alpha)^2+\beta^2\right)^i} = \frac{A_i(s-\alpha)+B_i\beta}{\left((s-\alpha)^2+\beta^2\right)^i},$$

where $1 \leq i \leq m$.

The calculation of the inverse Laplace transform in this case is more involved. It can be done as a combination of the property of the derivative of Laplace transform and the notion of *convolution* that will be discussed in section 6.6 or using decomposition to linear factors using complex roots as in Enrichment 8.