Some supporting material for the topic of matrix exponential, repeated eigenvalues and generalized eigenvectors

In this notes I present a slightly different point of view (then the one that was presented in class) on the topic how to find general solution of linear systems $x' = Ax$ in presence of repeated eigenvalues and when for some eigenvalues their geometric multiplicity is strictly smaller than the algebraic multiplicity

In this point of view it is possible to avoid (at least explicitely) of using the matrix which represents a given matrix in a given basis and the Jordan blocks (and their exponentials) (of course, all this is used implicitely but this way may look more compact) Hopefully, these notes may improve your understanding of the topic and clarify why the notion of generalized eigenvector arises naturally.

Recall several facts discussed in class

1) $e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots$;

2) $e^{tA}x^0$ is a solution of $\begin{cases} x' = Ax \\ x(0) = x^0 \end{cases}$ (1) ;

3) e^{tA} is a fundamental matrix of $x' = Ax$;

4) If $AB = BA$ then $e^{A+B} = e^A e^B$,

In particular, since the matrix λI commutes with any other matrix, then using the decomposition

$$A = \underbrace{\lambda I}_{A} + \underbrace{(A - \lambda I)}_{B} \quad \text{we get}$$

$$e^{tA} = e^{t\lambda I} e^{t(A-\lambda I)} = e^{\lambda t} I e^{t(A-\lambda I)} =$$

$$= e^{t\lambda} e^{t(A-\lambda I)}, \text{ i.e. } \boxed{e^{tA} = e^{\lambda t} e^{t(A-\lambda I)} \quad (2)}$$

Although, at the first glance, fact 2 provides an explicit solution for the IVP (1)

In practice it is hard to calculate $e^{tA}x_0$ directly by definition, because

$$e^{tA}x^0 = \left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots\right)x^0 =$$

$$= x^0 + tAx^0 + \frac{t^2}{2}A^2x^0 + \frac{t^3}{3!}A^3x^0 + \dots$$

is an <u>infinite sum</u>.

However for some special vectors x^0 $e^{tA}x^0$ can be calculated explicitely using formula (2)

i) For example, if $x^0 = v$ is an eigenvector of A corresponding to the eigenvalue λ then $(A-\lambda I)v = 0 \Rightarrow$ using formula (2)

$$e^{tA}v = e^{\lambda t}e^{t(A-\lambda I)}v = e^{\lambda t}\Big(v + t\underbrace{(A-\lambda I)v}_{0} + \underbrace{\dots}_{\text{the rest is zero}}\Big)$$

$= e^{\lambda t}v$, i.e. $e^{tA}v = e^{\lambda t}v$ (as we already know very well)

ii) If $x^0 = w$ is a generalized eigenvector of order 2 of A corresponding to an eigenvalue λ then $(A-\lambda I)^2 w = 0 \Rightarrow$

$$e^{tA} w = e^{\lambda t} e^{t(A-\lambda I)} w = e^{\lambda t}\Big(w + t(A-\lambda I)w + \frac{t^2}{2}\underbrace{(A-\lambda I)^2 w}_{0} + \underbrace{\cdots}_{\text{the rest is } 0}\Big) = e^{\lambda t}\big(w + t(A-\lambda I)w\big)$$

i.e $\quad e^{tA} w = e^{\lambda t}\big(w + t(A-\lambda I)w\big)$

(iii) More generally, if w is a generalized eigenvector of order k of A corresponding to an eigenvalue λ then $(A-\lambda I)^k w = 0 \Rightarrow$

$$e^{tA} w = e^{\lambda t} e^{t(A-\lambda I)} w = e^{\lambda t}\Big(w + t(A-\lambda I)w + \frac{t^2}{2}(A-\lambda I)^2 w + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1} w + \frac{t^k}{k!}\underbrace{(A-\lambda I)^k w}_{0} + \underbrace{\quad}_{\text{the rest is } 0}\Big) =$$

$$= e^{\lambda t}\underbrace{\Big(w + t(A-\lambda I)w + \frac{t^2}{2}(A-\lambda I)^2 w + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1} w\Big)}_{\text{finite number of terms}} \quad (3)$$

So, for a generalized eigenvector w, $e^{tA}w$ can be calculated explicitely by summing up a finite number of terms.

So, if we may choose a basis of generalized eigenvectors $w^1, w^2, \dots, w^n$ (some of them are eigenvectors, some are generalized eigenvectors of order 2 (if there is no a basis of eigenvector), some are generalized eigenvectors of order 3 (if there is no a basis of eigenvectors and generalized eigenvectors of order 2) etc) then

$$e^{tA}w^1, e^{tA}w^2, \dots, e^{tA}w^n$$ constitute a fundamental set of solutions and each term $e^{tA}w^i$ can be calculate by finite number of operations.

Two extremely useful fact from Linear Algebra helping to implement this idea are:

1) For any matrix A one can choose a basis of generalized eigenvectors

2) This basis may be chosen such that it consists of (may be several) chains $w, (A-\lambda I)w, \dots (A-\lambda I)^{k-1}w$ for some generalized eigenvector of order k

which is not of order $k-1$ (i.e $(A-\lambda I)^k w = 0$ but $(A-\lambda I)^{k-1} w \neq 0$). Then $(A-\lambda I)w$ is a generalized eigenvector of order $k-1$, which is not of order $k-2$ etc

If we denote $w^1 = w$, $w^2 = (A-\lambda I)w$, $w^3 = (A-\lambda I)^2 w$, ..

$$w^k = (A-\lambda I)^{k-1} w, \text{ then}$$

$$e^{tA} w^1 = e^{\lambda t}\Big(\underbrace{w}_{w^1} + t\underbrace{(A-\lambda I)w}_{w^2} + \frac{t^2}{2}\underbrace{(A-\lambda I)^2 w}_{w^3} + \dots + \frac{t^{k-1}}{(k-1)!}\underbrace{(A-\lambda I)^{k-1} w}_{w^k}\Big) = e^{\lambda t}\Big(w^1 + t w^2 + \frac{t^2}{2} w^3 + \dots + \frac{t^{k-1}}{(k-1)!} w^k\Big)$$

$$e^{tA} w^2 = e^{tA}(A-\lambda I)w^1 = (A-\lambda I)e^{tA} w^1 =$$

$$e^{\lambda t}(A-\lambda I)\Big(w^1 + t w^2 + \dots \frac{t^{k-1}}{(k-1)!} w^k\Big) = e^{\lambda t}\Big(\underbrace{(A-\lambda I)w^1}_{w^2} + t\underbrace{(A-\lambda I)w^2}_{w^3} + \dots + \frac{t^{k-2}}{(k-2)!}\underbrace{(A-\lambda I)w^{k-1}}_{w^{k-2}} + \frac{t^{k-1}}{(k-1)!}\underbrace{(A-\lambda I)w^k}_{0}\Big) =$$

$$= e^{\lambda t}\Big(w^2 + t w^3 + \dots + \frac{t^{k-2}}{k-2} w^{k-2}\Big) \text{ and so on.}$$