Phase portrait for $n=2$ in the case of complex eigenvalues

Let $\lambda = \alpha + i\beta$ be a complex eigenvalue of A with $\beta > 0$ and $a + ib$ be the correspon-ding eigenvector.

$$e^{(\alpha+i\beta)t}(a+ib) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(a+ib) =$$
$$= e^{\alpha t}(\cos\beta t\, a - \sin\beta t\, b) + i e^{\alpha t}(\sin\beta t\, a + \cos\beta t\, b) \Rightarrow$$

The general solution is

$$X(t) = c_1 e^{\alpha t}(\cos\beta t\, a - \sin\beta t\, b) + c_2 e^{\alpha t}(\sin\beta t\, a + \cos\beta t\, b) =$$
$$= \underbrace{e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)}_{x(t)} a + \underbrace{e^{\alpha t}(-c_1\sin\beta t + c_2\cos\beta t)}_{y(t)} b$$

In the coordinates corresponding to the basis (a, b) the trajectories are given by the following parametric equations: $x(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$

$$y(t) = e^{\alpha t}(-c_1\sin\beta t + c_2\cos\beta t)$$

Let $R = \sqrt{c_1^2 + c_2^2}$ and δ is such that $\cos\delta = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$, $\sin\delta = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$

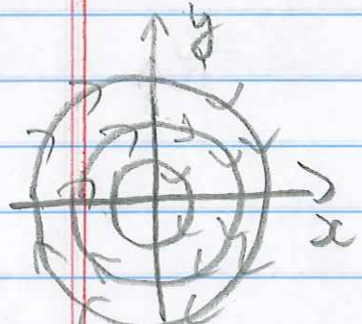
$$\Rightarrow x(t) = e^{\alpha t} R(\cos\delta\cos\beta t + \sin\delta\sin\beta t) = e^{\alpha t} R\cos(\beta t - \delta)$$
$$y(t) = e^{\alpha t} R(-\cos\delta\sin\beta t + \sin\delta\cos\beta t) = -e^{\alpha t} R\sin(\beta t - \delta)$$

$$\begin{cases} x(t) = e^{\alpha t} R\cos(\beta t - \delta) \\ y(t) = - e^{\alpha t} R \sin(\beta t - \delta) \end{cases}$$

<u>Case 1</u> $\alpha = 0$ $(\Leftrightarrow \operatorname{Re} \lambda = 0)$

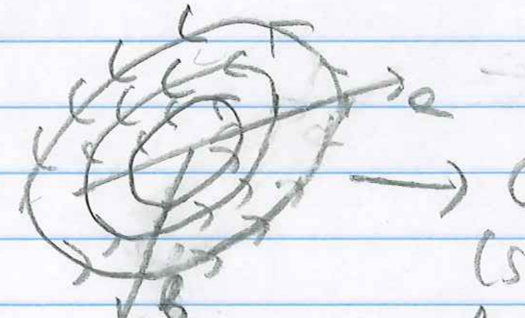
$$\begin{aligned} x(t) &= R\cos(\beta t - \delta) \\ y(t) &= -R\sin(\beta t - \delta) \end{aligned} \longrightarrow \text{circles } x(t)^2 + y^2(t) = R^2$$

in xy-plane with clockwise rotation



In original coordinates – ellipces

or



$\longrightarrow$ center (stable)

$\perp$

direction of motion $\to$ from b to a in the shortest way

<u>Case 2</u> $\alpha < 0$ $\to$ exponentially decaying factor $e^{\alpha t}$ $\Rightarrow$ circles becomes spirals

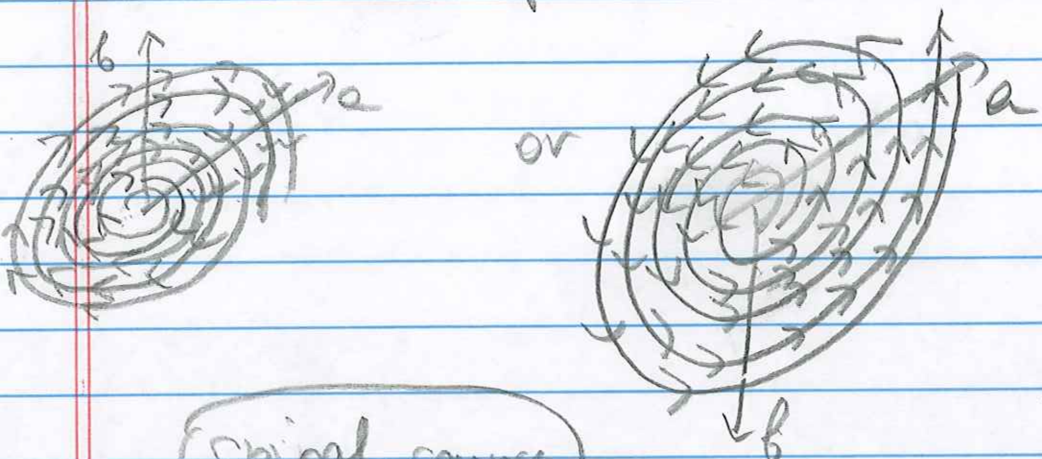
or

$\to$ spiral sink (stable) $\leftarrow$

$\to$ spirals approach origin.

Case 3 $\alpha > 0$ → exponentially increasing factor $e^{\alpha t}$

⇒ circles becomes spirals



spiral source

(unstable) → spirals go away from the origin

Remark Note that the directions of a and b have no geometric meaning, but (a, b) defines the orientation and it defines the direction of rotation (from b to a in the shortest way).

Rem From spiral sink to nodal sink through improper node (passing through critical damping)

Eigenvalues