MATH 309 PDE part

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Lecture 1 (end of class of 04/08 - class of 04/10)

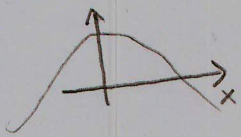
I Some basic PDE's

1. Wave equation / vibration of a string

(1) $$u_{tt} - a^2 u_{xx} = f(x,t),$$ where

$x \mapsto u(x,t)$ is a shape of the string/wave at time t

$a = \sqrt{\dfrac{T_0}{\rho(x)}}$, T_0 is a constant tension in the string; ρ is the mass density



or a is the velocity of sound

$f(x,t)$ is the force density

The derivation is based on the second Newton law. In the derivation of this equation many assumption are used to simplify the situation such that u_x^2 is neglected (assumed to be very small), the vibration of each point is in the direction perpendicular to x-axis. So, equation (1) describes some idealized vibration/wave propagation

Usually (as in ODE's) we have some initial/boundary conditions: Typically for a finite sping we have

$u(x,0) = \varphi(x)$ - initial shape/profile of the wave/spring

$u_t(x,0) = \psi(x)$ - initial velocity of the vibration prescribed at any point x

$u(0,t) = u(L,t)$ - this means that the string is kept at the endpoints

Similarly for vibrating membrane (drum)

$u_{tt} = a^2 \underbrace{\Delta u}_{\text{Laplacian}}$ $\quad t>0,\ x \in D$

$\Delta u = u_{xx} + u_{yy}$ (or $u_{xx} + u_{yy} + u_{zz}$

if we have 3 space variables)

Similar initial / boundary conditions:

$u(x,y,0) = \varphi(x,y)$ $\quad \forall (x,y) \in D$

$u_t(x,y,0) = \psi(x,y)$

$u(x,y,t) = 0$ $\quad \forall t \geq 0$ $(x,y) \in \partial D$, i.e the membrane

∂D: the boundary of D

or drum is glued to the boundary

2. <u>Heat or diffusion equation</u>

$u_t = k \dfrac{\partial^2 u}{\partial x^2}$ or $u_t = k \Delta u$

in 1-dim space

$u_t = k \Delta u$ in higher dimensional space

$u(x,t)$ is the temperature at a point x at time t

or the concentration of gas at a point x at time t

Derivation is based on the Fourier law that the heat (or gas) is spread in the direction of $-\nabla u$

∇u: gradient of u

Recall that $-\nabla u$ is exactly the direction in which u <u>decrease</u> in the maximal rate. There are two ways to calculate the amount of the heat or gas going out of a region V in a unit of time. One way is to take the derivative w.r.t. time of the total heat/gas in V, i.e (if we are in 3D) to calculate

③

$$-\frac{\partial}{\partial t}\iiint_V u(x,t)\,dV \qquad x=(x_1,x_2,x_3)$$

The other way is to calculate the amount of heat / gas going through the boundary ∂V (in the outside direction) and this is the flux: $\iint_{\partial V} -k\nabla u\cdot n\,dS$, where k is certain coefficient (diffusivity, depending on the space coordinates x in general)

$$\Downarrow$$

$$\frac{\partial}{\partial t}\iiint_V u(x,t)\,dV = \iint_{\partial V} k\nabla u\cdot n\,dS$$

On the other hand, using the Divergence theorem

$$\iint_{\partial V} k\nabla u\cdot n\,dS = \iiint_V \operatorname{div}(k\nabla u)\,dV \qquad (*)$$

Also $$\frac{\partial}{\partial t}\iiint_V u(x,t)\,dV = \iiint_V \frac{\partial}{\partial t}u(x,t)\,dV \qquad (**)$$

Shrinking V to a point and using the Mean Value Thm for integrals we get from $(*)$ and $(**)$:

$$\frac{\partial}{\partial t}u = \operatorname{div}(k\nabla u)$$

Now assume that k is constant $\Rightarrow$ r.h.s. $= k\operatorname{div}(\nabla u) = k\Delta u$

$$\left(\operatorname{div}(\nabla u) = \operatorname{div}(u_x, u_y, u_z) = u_{xx}+u_{yy}+u_{zz} = \Delta u\right)$$

Finally, we get $\boxed{u_t = k\Delta u}$ (2)

Initial / boundary conditions

$u(x,0)=\varphi(x)$ - the initial temperature / concentration

Rem u_t is presribed by the equation $u_t = k\Delta u$ $\Rightarrow$ there is no sense to prescribe u_t at $t=0$ as was done for the wave equation.

Conditions at the endpoints:

Example 1 $u(0,t)=u(L,t)=0$ for all $t>0$ — the meaning is that the endpoints are kept with 0 temperature

Example 2 $u_x(0,t)=u_x(L,t)=0$ for all $t>0$ — insulated endpoints: the heat does not pass through the end.

3. Laplace / Poisson equation

$$\Delta u = 0 \quad (3)$$

One appearance

Steady state of the heat equation

If u is a solution of the heat equatoin indepent of time ($\Leftrightarrow u_t=0$) then u (as a function of space variables) satisfies (3)

Such u is called steady state.

In other words (3) is the equation for steady state distribution of temperature.

(5) Another appearance is electrostatics as the equation for the potential of electric field of the given charge distribution.

Based on the Gauss Law:

If E is electric field then

$$\operatorname{div} E = \frac{\rho}{\varepsilon}$$, where ρ is the charge density, ε is a permitivity of the medium

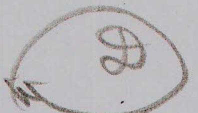
Then if u is the electric potential, $\nabla u = -E \Rightarrow$

$$\operatorname{div}(\nabla u) = -\frac{\rho}{\varepsilon} \Leftrightarrow \boxed{\Delta u = -\frac{\rho}{\varepsilon}}$$

(see discussions of heat eq)

$\Delta u = f$ is called Poisson equation

Typical boundary conditions: $u\big|_{\partial \mathcal{D}} = \varphi$ — Dirichlet conditions



or $\frac{\partial u}{\partial n}\Big|_{\partial \mathcal{D}} = \psi$ — Neumann conditions

derivative in the direction of the outer normal

II Classification of linear systems of PDE's of second order

All equations above in the case of two independent variables x and y are of the form

$$A(x,y)\, u_{xx} + B(x,y)\, u_{xy} + C(x,y)\, u_{yy} + D(x,y)\, u_x + E(x,y)\, u_y + F(x,y)\, u = G(x,y) \qquad (4)$$

Let $L(u) = A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y +$
$+ F(x,y)u$.

Then L is a linear transformation (of infinite dimensional space of functions), because

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$$

for any twice differentiable functions u_1 and u_2 and any constants c_1 and c_2.

So, one can rewrite the eq. (4) (see the previous page) as $L(u) = G$

and look on this as infinite dimensional analog of systems of linear equations we discuss in the Linear Algebra part of the course.

Classification There are tree types of second order PDE's depending on the sign of the discriminant

$$B^2 - 4AC$$

1) If $B^2-4AC>0$ at a point (x,y) then the eq. (4) is called hyperbolic at a point (x,y)
2) If $B^2-4AC=0$ at a point (x,y) then the eq. (4) is called parabolic at a point (x,y)
3) If $B^2-4AC<0$ at a point (x,y) then the eq (4) is called elliptic at a point (x,y)

It turns out that equations of different types have different properties, in particular the boundary value problems are defined for them in a different way. This is also related to the notion of characteristics that we will not discuss in our course

Rem In general, if A, B, C are not constant, the same equation might be of different type at different points

For example $u_{xx} + y u_{yy} = 0$ is elliptic if $y > 0$, parabolic if $y = 0$ and hyperbolic if $y < 0$.

Rem Note that the type depends on the coefficients near highest derivatives only

Let us examine what type has each basic equation in this classification.

1) Wave equation $u_{xx} - a^2 u_{tt} = \ldots$

Let x play role of x and t play role of y in (4) of page 5 (actually it is not important in what order we will take the independent variables, the order does not change the type)

Then $A = 1$, $B = 0$, $C = -a^2 \Rightarrow$

$$B^2 - 4AC = 0 - (-4a^2) = 4a^2 > 0 \Rightarrow$$ the wave equation is hyperbolic (at any point)

2) Heat equation $u_t = k u_{xx} \Leftrightarrow k u_{xx} \underbrace{- u_t}_{\text{this is of order 1 so it does not effect the type}} = 0$

Again if x plays role of x and t play role of y in (4) of page 5

Then $A = k$, $B = 0$, $C = 0 \Rightarrow$

$B^2 - 4AC = 0 \Rightarrow$ the heat equation is parabolic (at any point)

3) Laplace $u_{xx} + u_{yy} = 0$

$A = 1$; $B = 0$, $C = 1 \Rightarrow B^2 - 4AC = 0 - 4 < 0 \Rightarrow$ the Laplace equation is elliptic at any point.

<u>Rem</u> (enrichment) Similar classification can be defined for second order linear PDE on functions of number of independent variables >2. This topic is closely related to the theory of quadratic forms (the classical topic in Linear Algebra that, unfortunately, is not in the syllabus of our course, see section 6.6 of the Leon book)

<u>Rem</u> (enrichment) In the case of 2 independent variables one can show that for any hyperbolic equation in a neighbourhood of any point (x_0, y_0) one can find a change of (independent) variables

$$x = x(\tilde{x}, \tilde{y})$$
$$y = y(\tilde{x}, \tilde{y})$$

such that in the new coordinates the equation has the form

$$\underbrace{u_{\tilde{x}\tilde{x}} - u_{\tilde{y}\tilde{y}}}_{\substack{\text{principle}\\\text{part is}\\\text{the way equation}}} + \underbrace{\cdots}_{\substack{\text{terms depending}\\\text{on } u_{\tilde{x}},\, u_{\tilde{y}} \text{ and } u}} = 0$$

Similar statement can be formulated for parabolic versus heat eq. and elliptic versus Laplace eq. It shows that the consideration of the wave, heat and Laplace equation is not occasional and can help in understanding more general equations.