MATH 309 PDE part   Igor Zelenko

Lecture 2 (class of 04/12)

[In the first 15 minutes we discussed the typical mistakes on midterm #2

I Examples of solutions of some PDE
Sketch of D'alambert method for wave equation.

General solution of ODE depend on several arbitrary constants.

General solution of PDE depend on arbitrary functions

Example 1 Find the general solution of $u_x = \sin(2x+y)$

(you already dealt with this type of equations in the "Multivariable Calculus" MATH 251, when studying how to find a potential of a conservative vector field or in "Differential Equation", MATH 308, when studying how to solve exact equations)

Solution $u_x = \sin(2x+y) \Rightarrow u = \int \sin(2x+y)\,dx + \underbrace{C(y)}_{\text{an arbitrary function of } y} =$

$$= \boxed{-\frac{\cos(2x+y)}{2} + C(y)}$$

Example 2 Find the general solution of the equation

$$u_{xy} = 0$$

Solution

$u_{xy}=0 \Leftrightarrow (u_x)_y=0 \underset{\text{as in example 1}}{\Rightarrow} u_x = \underbrace{h(x)}_{\text{an arbitrary function of } x} \Rightarrow$

$$u = \underbrace{\int h(x)\,dx}_{f(x)} + g(y) = \boxed{f(x)+g(y)}$$, i.e. is sum of an arbitrary function of x and an arbitrary function of y.

Example 3 (wave equation for infinite spring) d'Alembert method, sketch

Consider the wave equation

$$u_{tt} - a^2 u_{xx} = 0 \qquad (1)$$

Make the following change of coordinates

$$\begin{aligned} \tilde{x} &= x+at \\ \tilde{y} &= x-at \end{aligned} \quad \xRightarrow{\text{Chain rule}} \quad \begin{aligned} u_x &= u_{\tilde{x}} + u_{\tilde{y}} \\ u_t &= a u_{\tilde{x}} - a u_{\tilde{y}} \end{aligned} \quad \Rightarrow$$

$$u_{xx} = u_{\tilde{x}\tilde{x}} + u_{\tilde{x}\tilde{y}} + u_{\tilde{y}\tilde{x}} + u_{\tilde{y}\tilde{y}} = u_{\tilde{x}\tilde{x}} + 2u_{\tilde{x}\tilde{y}} + u_{\tilde{y}\tilde{y}}$$

$$u_{tt} = a^2 u_{\tilde{x}\tilde{y}} - a^2 u_{\tilde{x}\tilde{y}} - a^2 u_{\tilde{y}\tilde{x}} + a^2 u_{\tilde{y}\tilde{y}} = a^2(u_{\tilde{x}\tilde{y}} - 2u_{\tilde{x}\tilde{y}} + u_{\tilde{y}\tilde{y}})$$

$\Downarrow$

$$u_{tt} - a^2 u_{xx} = \cancel{a^2 u_{\tilde{x}\tilde{x}}} - 2a^2 u_{\tilde{x}\tilde{y}} + \cancel{a^2 u_{\tilde{y}\tilde{y}}} - \cancel{a^2 u_{\tilde{x}\tilde{x}}} - 2a^2 u_{\tilde{x}\tilde{y}} - \cancel{a^2 u_{\tilde{y}\tilde{y}}}$$

$= -4a^2 u_{\tilde{x}\tilde{y}} \Rightarrow$ In the new coordinate the original equation (1) has the form $u_{\tilde{x}\tilde{y}} = 0$, i.e. exactly as in the previous example 1

$$\Rightarrow u = f(\xi) + g(\eta) = f(x+at) + g(x-at)$$

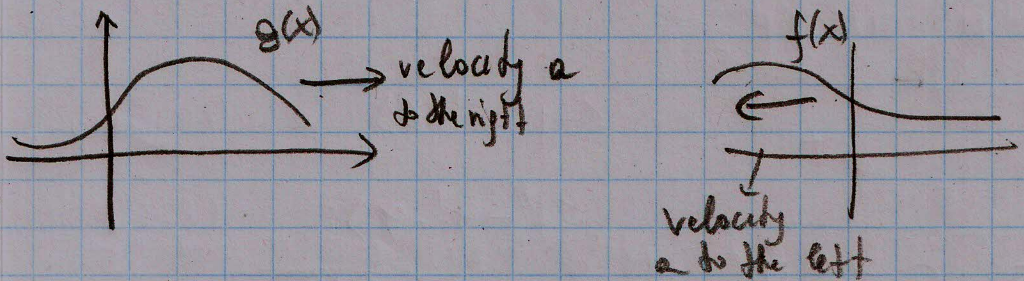
returning to the original variables

$\Rightarrow$ the general solution of $u_{tt} - a^2 u_{xx} = 0$ is

(2) $\boxed{u(x,t) = f(x+at) + g(x-at)}$, for arbitrary functions f and g of single variable.

What is the physical meaning of this solution?

The graph of function $g(x-at)$ for given t is obtained from the graph of function $g(x)$ by a shift in at to the left. In other words, imagine that we start with the wave (or spring) with the shape $g(x)$. Then the family of graphs of functions $g(x-at)$ is obtained when the graph of function $g(x)$ moves to the right with constant velocity a $\rightarrow$ travelling wave.



In the same way, the family of graphs of functions $f(x+at)$ is obtained by moving of the graph of function $f(x)$ to the left with constant velocity a $\rightarrow$ another travelling wave

Thus, the general solution of (1) is a <u>superposition of two travelling waves</u>

II <u>Enrichment</u> (sketch of D'alamber method)

If we have initial shape of the wave and the initial velocity (and we consider an infinite string) i.e

$$u(x,0) = \varphi(x) \qquad (3)$$

$$u_t(x,0) = \psi(x) \qquad (4)$$

then the functions f and g can be easily expressed via the initial conditions φ and ψ

Namely, if $u(x,t) = f(x+at) + g(x-at) \Rightarrow$

$$u(x,0) = f(x) + g(x) \overset{(3)}{=} \varphi(x) \qquad (5)$$

$$u_t(x,0) \underset{\substack{\text{chain}\\ \text{rule}}}{=} a f'(x) - a g'(x) = \psi(x) \qquad (6)$$

Differentiating (5) we get $f'(x) + g'(x) = \varphi'(x)$ i.e the system of 2 equations for $f'(x)$ and $g'(x)$

$$\begin{cases} f'(x) + g'(x) = \varphi'(x) \\ f'(x) - g'(x) = \frac{1}{a}\psi(x) \end{cases} \Rightarrow \text{we can find } f'(x) \text{ and}$$

$g'(x) \Rightarrow$ we can find $f(x)$ and $g(x)$

(note that $f(x)$ and $g(x)$ are defined up to constants

This material will not be asked in the homework or exams. This is enrichment for your better understanding.

Enrichment

But the condition that $f(x)+g(x) = \varphi(x)$ will define the solution of the form $f(x+at)+g(x-at)$ uniquely)

<u>Rem</u> In the case of the finite string, i.e

when $\begin{cases} u(x,0) = \varphi(x) & 0 \le x \le L \\ u_t(x,0) = \psi(x) & \\ u(0,t) = 0 & t>0 \\ u(L,t) = 0 & \end{cases}$

one can still use the method above by defining the initial shape φ and ψ on the whole $\mathbb{R}$ via a series of reflections (physical meaning is that the solution is the superposition of travelling waves and the waves obtained by relfections of the travelling waves at the endpoints

There are many different methods of solving PDE (the method of characteristic, separation of variables, Fourier transform etc). In the rest of the course we will focus on one method, the method of separation of variables.

III The method of separation of variables for PDE's

(the main idea)

Rem Not that although it has the same name as a method in ODE, but in general these are different methods.

We will describe an idea via an example

Example

Solve $u_t = 9u_{xx}$ (7)

on $0 < x < 4$, $t > 0$ with initial conditions

$$u(x,0) = 2\sin\frac{\pi}{2}x - 3\sin 2\pi x$$

$$u(0,t) = u(4,t) = 0$$

and the additional assumption that $u(x,t)$ is bounded on the region we are looking at.

The idea Let us first look for the solutions in the form

(8) $u(x,t) = X(x)T(t)$, i.e as a product of two functions of single variables.

Substitute (8) into equation $\Rightarrow$

$$XT' = 9X''T \Rightarrow \underbrace{\frac{T'}{9T}}_{\text{this part depends on } t \text{ only}} = \underbrace{\frac{X''}{X}}_{\text{this part depend on } x \text{ only}} = \lambda \Rightarrow$$

$\lambda = \frac{T'}{9T} \Rightarrow \lambda$ does not depend on x; $\lambda = \frac{X''}{X} \Rightarrow \lambda$ does not depend on t

$\Rightarrow \boxed{\lambda \text{ is constant}} \Rightarrow$ Instead of one PDE (7) we arrived to 2 ODE's: $\begin{cases} T' = 9\lambda T \\ X'' = \lambda X \end{cases}$ for some constant λ.

Will be continued on 04/15