MATH 309 PDE part Igor Zelenko

Lecture 3 (class 04/15)

The method of separation of variables for PDE's

(the main idea)

Last time we started the following example

Example Solve $u_t = 9u_{xx}$ (1)

on $0<x<4$, $t>0$ with initial/boundary conditions

$$u(x,0) = 2\sin\frac{\pi}{2}x - 3\sin 2\pi x \quad (2)$$

$$u(0,t) = u(4,t) = 0 \quad (3)$$

Note that during the class we made for simplicity an additional assumption that $u(x,t)$ is bounded on the region we are looking at, i.e. there exist a constant M such that $|u(x,t)|<M$ for $1<x<4$, $t>0$

But this assumption can be removed.

Solution First look for the solutions of (1)

of the form (4) $u(x,t) = X(x)T(t)$, i.e. a solution which is a product of two functions of single variables.

Substitute (4) into equation (1):

$$XT' = 9X''T \Rightarrow \frac{T'}{9T} = \frac{X''}{X} = C \Rightarrow$$

$$\underbrace{\frac{T'}{9T}}_{\substack{\text{this part}\\ \text{depends}\\ \text{on } t \text{ only}}} = \underbrace{\frac{X''}{X}}_{\substack{\text{this part}\\ \text{depends}\\ \text{on } x \text{ only}}} = c$$

$$\left.\begin{array}{l} c = \dfrac{T'}{9T} \Rightarrow c \text{ does not depend on } x \\ c = \dfrac{X''}{X} \Rightarrow c \text{ does not depend on } t \end{array}\right\} \Rightarrow c \text{ is a constant}$$

$\Rightarrow$ we have two separate ordinary differential equations:

$$T' = 9cT \qquad (5)$$
$$X'' = cX \qquad (6)$$

Now use the boundary conditions (3), i.e

$u(0,t) = u(4,t) = 0 \quad \forall t > 0 \Rightarrow$

$X(0)\,T(t) = X(4)\,T(t) = 0$

We are looking for nonzero solutions $\Rightarrow T(t) \neq 0 \Rightarrow$

$$X(0) = X(4) = 0 \qquad (3')$$

Let us show first that $\underline{c < 0}$

① If $c > 0$ then $X'' - cX = 0 \Rightarrow$ char eq. is $r^2 - c = 0 \Rightarrow r_{1,2} = \pm\sqrt{c}$

$$X = Ae^{\sqrt{c}x} + Be^{-\sqrt{c}x}$$

$X(0) = 0 \Rightarrow A + B = 0$, $X(4) = 0 \Rightarrow Ae^{4\sqrt{c}} + Be^{-4\sqrt{c}} = 0$

$\Rightarrow$ we have the following linear system of 2 eq. for A & B

$$\begin{cases} A+B=0 \\ e^{4\sqrt{c}}A+e^{-4\sqrt{c}}B=0 \end{cases}$$

The determinant of the system is

$$\begin{vmatrix} 1 & 1 \\ e^{4\sqrt{c}} & e^{-4\sqrt{c}} \end{vmatrix} = e^{-4\sqrt{c}}-e^{4\sqrt{c}} < 0 \Rightarrow A=B=0$$

$X(x)\equiv 0$ but we are looking for a nonzero solutons.

② If $c=0 \Rightarrow X''=0 \Rightarrow X(x)=A+Bx \Rightarrow$ again $X(0)=X(4)=0 \Rightarrow$

implies that $A=B=0$.

So, $c<0 \Rightarrow \boxed{c=-\lambda^2}$ for some $\lambda>0$

(6) $\Rightarrow X''+\lambda^2X=0 \Rightarrow X(x)=B\cos\lambda x+C\sin\lambda x$

The boundary conditions impose additional restrictions on λ:

$X(0)=0 \Rightarrow B=0$

$X(4)=0 \Rightarrow C\sin 4\lambda=0 \Rightarrow 4\lambda=\pi n$, $n\in \mathbb{N}$ (not that n and $-n$ give the same solution up to a constant)

$\lambda=\frac{\pi n}{4}$

$\Rightarrow X(x)=C\sin\frac{\pi n}{4}x \quad n\in\mathbb{N}$

On the other hand, equation (5) has the form (with $c=-\lambda^2=-\frac{\pi^2n^2}{16}$)

$$T'=-\frac{9\pi^2n^2}{16}T \Rightarrow T=Ae^{-\frac{9\pi^2n^2}{16}t} \Rightarrow$$

If $u(x,t)=X(x)T(t)$ is a solution of (1) satisfying boundary conditions (3) then $u(x,t)=Ce^{-\frac{9\pi^2n^2}{16}t}\sin\frac{\pi n}{4}x$ for some natural n and constant C

$$\boxed{u(x,t) = Ce^{-\frac{9\pi^2 n^2}{16}t} \sin\frac{\pi n}{4}x, \quad n \in \mathbb{N}} \quad (7)$$

Now we have to satisfy the initial conditions (2)

Since eq (1) and boundary conditions (3) are homogeneous

the superposition principle holds $\Rightarrow$ we try to find

the solution of (1), satisfying (2) and (3) as a superposition

(= sum) of the solutions of (7)

Namely, $u(x,t) = \sum_{n=1}^{N} C_n e^{-\frac{9\pi^2 n^2}{16}t} \sin\frac{\pi n}{4}x$ for some N is a solution

of (1) satisfying (3) but we also want that

it will satisfy the boundary conditions (2), i.e

$$u(x,0) = \sum_{n=1}^{N} C_n \sin\frac{\pi n}{4}x = \underbrace{2\sin\frac{\pi}{2}x}_{\text{corresponds to } n=2} - \underbrace{3\sin 2\pi x}_{\text{corresponds to } n=8}$$

$\Downarrow$

$C_2 = 2$, $C_8 = -3$ and all others C_i are $0 \Rightarrow$

The required solutions is

$$u(x,t) = 2e^{-\frac{9\pi^2 \cdot 4t}{16}} \sin\frac{\pi}{2}x - 3e^{-\frac{9\pi^2 \cdot 64t}{16}} \sin 2\pi x =$$

$$= \boxed{2e^{-\frac{9\pi^2 t}{4}} \sin\frac{\pi}{2}x - 3e^{-36\pi^2 t} \sin 2\pi x}$$

In the previous example the initial conditions was quite special (i.e. trigonometric polynomial satisfying the boundary conditions $u(0,t)=u(4,t)=0$)

If, instead, we have

$$u(x,0)=f(x) \qquad 0<x<4$$

then the natural idea is to represent f in the form

$$f(x) \approx \sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{4}$$

(the so-called <u>half-range Fourier series</u>),

if possible, and then the solution will be

$$\sum_{n=1}^{\infty} B_n e^{-\frac{9\pi^2 n^2}{16}t} \sin\frac{\pi n x}{4}$$

This motivates to develop the theory of Fourier series that will be done later.

Before this I will give few more examples on separability.

<u>Example 2</u> Does the equation

$$u_{xx}-2u_{yy}+3u_y=0 \text{ separates?}$$

If so, find the resulting ODE (we do not discuss any boundary conditions here)

Solution Let $u = X(x)\,Y(y)$ => (pluging into equation)

$$X''Y - 2XY'' + 3XY' = 0$$

Divide the equation by XY:

$$\frac{X''}{X} - 2\frac{Y''}{Y} + 3\frac{Y'}{Y} = 0 \Rightarrow$$ we can move the terms with X and Y to the different parts

$$\underbrace{\frac{X''}{X}}_{\text{depends on } x \text{ only}} = \underbrace{2\frac{Y''}{Y} - 3\frac{Y'}{Y}}_{\text{depends on } y \text{ only}} = \underbrace{\lambda}_{\text{constant}}$$ → the equation separates

$$\Rightarrow \begin{cases} X'' = \lambda X \\ 2Y'' - 3Y' - \lambda Y = 0 \end{cases} \rightarrow 2 \text{ ODE's}$$

Example (the case of functions of 3 variables)

Separate $\quad y u_{xx} + x u_{yy} + xy\, u_{zz} = 0$, finding the resulting ODE's.

Solution We look for $u(x,y,z)$ in the form

$$u(x,y,z) = X(x)\,Y(y)\,Z(z)$$

Substituting into equation we get

$$y\, X''YZ + x\, XY''Z + xy\, XYZ'' = 0$$

Divide the equation by $xy\,XYZ$:

$$\frac{X''}{xX} + \frac{Y''}{yY} + \frac{Z''}{Z} = 0 \Rightarrow \frac{X''}{xX} + \frac{Y''}{yY} = -\frac{Z''}{Z}$$ (see the next page)