7

$$\underbrace{\frac{X''}{xX}+\frac{Y''}{yY}}_{\text{depends on } x,y \text{ only}} = \underbrace{-\frac{Z''}{Z}}_{\text{depends on } z \text{ only}} = \underbrace{\lambda}_{\text{constant}}$$

$$\Rightarrow \quad Z''+\lambda Z=0$$

Further, $\frac{X''}{xX}+\frac{Y''}{yY}=\lambda \Rightarrow \underbrace{\frac{X''}{xX}}_{\text{depends on } x \text{ only}} = \underbrace{\lambda-\frac{Y''}{yY}}_{\text{depends on } y \text{ only}} = \underbrace{\mu}_{\text{constant}} \Rightarrow$

$$X''=\mu x X$$

$$\frac{Y''}{yY}=\lambda-\mu \Rightarrow Y''=(\lambda-\mu)yY$$

Thus we get 3 ODE's:

$$\begin{cases} Z''+\lambda Z=0 \\ X''=\mu x X \\ Y''=(\lambda-\mu)yY \end{cases}$$ depending on two constants λ and μ.

Fourier series and their applications for solving PDE's

As we already discussed after the first example, it might be useful to represent a function as a sum (in general, containing infinite many terms) of trigonometric functions with frequencies that are integer multiples of one frequency)

More precisely, let f be a function on the interval $[-L, L]$ such that f is (Riemann) integrable on $[-L, L]$ (actually the definition works for more general class of functions related to the notion of <u>Lebesgue</u> integral but we do not have tools to define it here)

The Fourier series corresponding to f on $[-L, L]$ is a series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad (8)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad (9)$$

((1) for $n=0$ reads: $a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx \Rightarrow \frac{a_0}{2}$ is the mean of f on $[-L, L]$)

<u>Remark/motivation</u> If f is continuous on $[-L, L]$ (for simplicity) as we have seen already the partial sum

$$S_N(f) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \qquad (10)$$

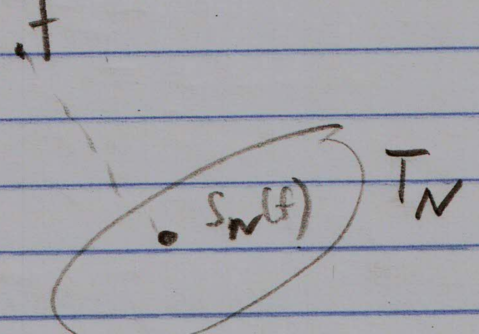
with coefficients a_n and b_n as in formules (8) and (9) above is the best approximation of f (the least square approximation) f among all trigonometric polynomials

$$\frac{\tilde{a}_0}{2} + \sum_{n=1}^{N} \tilde{a}_n \cos\frac{n\pi x}{L} + \tilde{b}_n \sin\frac{n\pi x}{L} \qquad (11)$$

w.r.t. the norm given by the inner product $(f,g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$ (12)

Geometric interpretation

Let T_N be the subspace of trigonometric polynomiel of the type (11) in $C([-L, L])$. Then $S_N(f)$ is exactly the orthogonal projection of the function f (the point in $C([-L, L])$) to the space T_N $\Rightarrow$



$S_N(f)$ is the closest point to f among all points of T_N in the norm of $C([-L, L])$ given by the the inner product (12).