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MATH 309 PDE part Igor Zelenko

Lecture 4 (class 04/17)

Elements of Fourier series

From the last class. Let f be a Riemann integrable function on $[-L, L]$ (more generally one can assume that f^2 is integrable in improper sense; since we have no the notion Lebesgue integral in our disposal, the assumptions on f here might look quite cumbersome). The Fourier series of f on the interval $(-L, L)$ is a series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \text{ where}$$

(1) $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad \left(\text{for } n=0 \text{ it reads } a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx\right)$$

(2) $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

The motivation for this definition is the least square approximation: If T_N is a subspace in $C[-L, L]$ spanned by the set of functions

(3) $$\left\{\frac{1}{\sqrt{2}}, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \dots, \cos\frac{N\pi x}{L}, \sin\frac{N\pi x}{L}\right\}$$

then the partial sum of the series, i.e

$$S_N(f) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

is exactly the orthogonal projection of f to the space T_N (w.r.t. the inner product

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(x) g(x)\, dx \,)$$

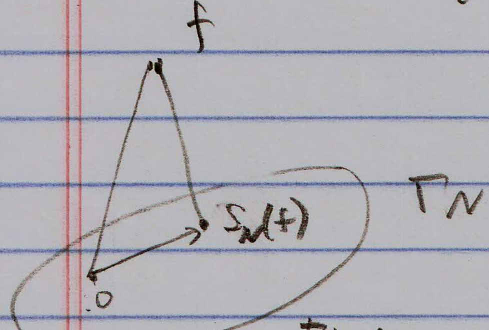

Fig 1

Therefore, by Pythagoras theorem $S_N(f)$ is the closest point to f in the subspace T_N (w.r.t. to the norm given by $\|f\| = \frac{1}{\sqrt{L}} \left(\int_{-L}^{L} (f(x))^2 dx \right)^{1/2}$)

Bessel inequality and Parseval's identity

Since $f - S_N(f) \perp T_N$, from the Pythagoras theorem (see Fig 1) it follows that

$$\|f - S_N(f)\|^2 + \|S_N(f)\|^2 = \|f\|^2 \quad (4)$$

$$\Downarrow$$

$$\|S_N(f)\|^2 \leq \|f\|^2 \quad (5)$$

Note that the set (3) on the previous page is orthonormal and $S_N(f) = \frac{a_0}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \sum_{n=1}^{N} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$.

In other words $S_N(f)$ have coordinates $\left(\frac{a_0}{\sqrt{2}}, a_1, b_1, a_2, b_2, \ldots, a_N, b_N \right)$ w.r.t. the orthonormal basis (3) of T_N $\Rightarrow$ by finite dim. Parseval's formula (see Leon, p. 243, Corollary 5.5.4)

$$\|S_N(f)\|^2 = \frac{a_0^2}{2} + \sum_{n=1}^{N}\left(a_n^2+b_n^2\right) \qquad (6)$$

$$\Downarrow (5)$$

$$\underbrace{\frac{a_0^2}{2} + \sum_{n=1}^{N}\left(a_n^2+b_n^2\right) \le \|f\|^2 \left(= \frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx \right)}$$

<u>Bessel's inequality</u>

With some additional work one can show that if f is Riemann integrable on $[-L,L]$ (or, more generally, integrable in an improper sense on $[-L,L]$, or, more generally, f^2 is Lebesgue integrable on $[-L,L]$) then

(7) $\|S_N(f)-f\| \xrightarrow[N\to\infty]{} 0$. In this case one says that the Fourier series <u>converge by mean</u> (or by square mean) to f

Then from (4) & (7) it follows that

$$\lim_{N\to\infty} \|S_N f\|^2 = \|f\|^2 \overset{(6)}{\Longrightarrow}$$

$$\boxed{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right) = \|f\|^2} \qquad (8)$$

This is called <u>Parseval's identity</u>

(4)

Parseval's identity is in a sense an infinite dimensional version of Phytagores thm.

Note that the question: in what sense the Fourier series of f converges to f should be clarified.

In general, there are many non-equivalent ways to define the notion of convergence in a functional space.

We already have seen one type of convergence, the convergence by mean

Another type of convergence is <u>pointwise convergence</u>.

The partial sum $S_N(f)$ is a function of x (a trigonometric polynomial). The questions are:

1) Given x does the sequence $\{S_N(f)(x)\}$ converge (as $N \to \infty$)?

2) And if yes, does it converge to $f(x)$?

These questions, in general, are very subtle.

We will briefly discuss the simplest aspects of these questions only

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For this, let me remind you several notions:
of periodic functions and piecewise continuous functions

Periodic functions

A function $f: \mathbb{R} \to \mathbb{R}$ is called periodic if there exists $T>0$ such that

$$f(x+T)=f(x) \qquad (9)$$

The number T is called a period of f

Note that 1) if $f(x) \equiv \text{const}$ then any $T>0$ is a period of f;

2) if $T>0$ is a period of f then nT is a period for any positive integer n

3) Assume that there exists the least period $T>0$ of f. Then any period $T_1>0$ of f is the integer multiple of T (i.e. $T_1 = nT$ for some positive integer n)

Rem In general, one can construct examples of nonconstant periodic functions for which there are no least (positive) period. (I do not give such examples here)

(6)

- For $\cos x$ and $\sin x$: all periods are $2\pi n$, $n \in \underbrace{N}_{\text{natural}}$

the least period is 2π

- For $\cos ax$ and $\sin(ax)$: all periods are $\frac{2\pi n}{a}$, the least period is $\frac{2\pi}{a}$

(For example $\cos\left(a\left(x+\frac{2\pi n}{a}\right)\right) = \cos(ax+2\pi n) = \cos(ax)$

i.e $f\left(x+\frac{2\pi n}{a}\right) = f(x) \Rightarrow \frac{2\pi n}{a}$ is a period

of $\cos x$. To show that there is no other periods

you can consider the equation

$$\cos(a(x+T)) = \cos ax \Leftrightarrow$$

$$\cos(a(x+T)) - \cos ax = 0 \Rightarrow$$

$$-2\sin\frac{a(x+T)-ax}{2}\sin\frac{a(x+T)+ax}{2} = 0 \Rightarrow$$

$$\sin\frac{aT}{2}\sin\left(ax+\frac{aT}{2}\right) = 0$$

The latter should hold for any $x \Rightarrow$

$$\sin\frac{aT}{2} = 0 \Rightarrow \frac{aT}{2} = \pi n \Rightarrow T = \frac{2\pi n}{a} \text{ q.e.d})$$

- So, all trigonometric polynomials appearing in as partial sums of Fourier series on $[-L, L]$

⑦

are periodic with period $\frac{2\pi}{\frac{\pi}{L}} = \boxed{2L}$

Therefore it make sense to extend the domain of a function f (which initially is defined on $(-L,L)$) to the whole $\mathbb{R}$ by periodicity, i.e by the rule

$$f(x+2L) = f(x)$$

(note, that I use the word domain here not in the sense you get used for: f might be not defined on some small (in certain sense) subset of $(-L,L)$)

In the sequel we will work from the beginning with periodic functions with period $2L$ and in this case it make sense we can speak on the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$ as a <u>Fourier series of f of period $2L$</u>.

Note also that in this case in the formulas (1) and (2) for the coefficients of Fourier

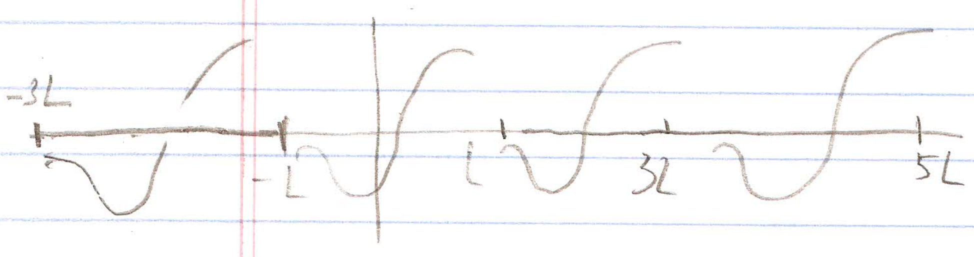
Series we can integrate not only over the interval $[-L, L]$ but also over any interval of the length $2L$, namely

$$a_n = \frac{1}{L} \int_{c-L}^{c+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad (10)$$

$$b_n = \frac{1}{L} \int_{c-L}^{c+L} f(x) \sin \frac{n\pi x}{L} \, dx \qquad (11)$$

for any c.

If we know the graph of f over $[-L, L]$ then to sketch the graph of its periodic extension to the whole $\mathbb{R}$ with period $2L$ you just shift this graph to the left and right by $2Ln$.



Piecewise continuous functions and Dirichlet condition

Def A function f is called piecewise continuous on the interval $[a, b]$ if f is continuous except (maybe) finite number of points $a = x_0 < x_1 < \ldots < x_n = b$