and for each x_i the one-sided limits

$$\lim_{x\to x_i-} f(x) \text{ and } \lim_{x\to x_i+} f(x) \text{ exist and finite}$$

(we will denote $f(x_i+) = \lim_{x\to x_i+} f(x)$
$f(x_i-) = \lim_{x\to x_i-} f(x)$)

<u>Def</u> One says that a function f satisfies the Dirichlet conditions if f and f' are piecewise continuous on $[-L, L]$ and f is periodic of period $2L$

<u>Thm</u> (Dirichlet) Assume that a function f satisfies Dirichlet conditions. Then the Fourier series of f (of period $2L$) converges to

(a) $f(x)$, if x is a point of continuity of f

(b) $\frac{f(x+)+f(x-)}{2}$, if x is a point of discontinuity

(In other words, $\lim_{N\to\infty} S_N(f)(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{f(x+)-f(x-)}{2} & \text{if } f \text{ is discontinuous at } x \end{cases}$

(Note that (b) ⇒ (a) if x is a point of continuity)

Example 1 a) Find the Fourier series of the function

$$f(x) = x \quad , -\pi < x \le \pi .$$

Solution
In our case $L = \pi \Rightarrow \frac{n\pi x}{L} = nx$

Calculate the coefficients a_n and b_n using formulas (1) & (2) of page 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \underset{\substack{\text{integration}\\ \text{by parts}}}{=} \frac{1}{\pi}\frac{x\sin nx}{n}\Big|_{-\pi}^{\pi} - \frac{1}{\pi n}\int_{-\pi}^{\pi}\sin nx\,dx =$$

$$= \frac{1}{\pi}\Big(\pi \underbrace{\sin n\pi}_{0} - \pi \underbrace{\sin(-n\pi)}_{0}\Big) + \frac{1}{\pi n^2}\cos nx\Big|_{-\pi}^{\pi} =$$

$$= \frac{1}{\pi n^2}\underbrace{\cos \pi n - \cos(-\pi n)}_{0} = 0$$

Rem We can make the same conclusions without the calculations above based on the fact that the functions $x\cos nx$ are odd i.e. $(-x)\cos(-nx) = -(x\cos nx)$
$\Rightarrow$ integrals of this functions over the intervals symmetric w.r.t. 0 are equal to zero

In particular $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx = 0$

Farther,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \underset{\substack{\text{integration} \\ \text{by parts}}}{=} - x\frac{\cos nx}{\pi n}\Big|_{-\pi}^{\pi} +$$

$u = x$	$v' = \sin nx$
$u' = 1$	$v = -\frac{\cos nx}{n}$

$$+ \frac{1}{\pi n}\int_{-\pi}^{\pi} \cos nx \, dx = \frac{-\pi \cos n\pi - \pi \overset{\cos n\pi}{\overbrace{\cos(-n\pi)}}}{\pi n} +$$

$$+ \underbrace{\frac{1}{\pi n^2} \sin nx \Big|_{-\pi}^{\pi}}_{0} = - \frac{2\cos \pi n}{n} = -\frac{2}{n}(-1)^n =$$

(here we use that $\cos \pi n = (-1)^n$)

$$= \boxed{\frac{2}{n}(-1)^{n+1}} \Rightarrow \boxed{b_n = \frac{(-1)^{n+1}2}{n}}$$

$\Rightarrow$ The Fourier series is $2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

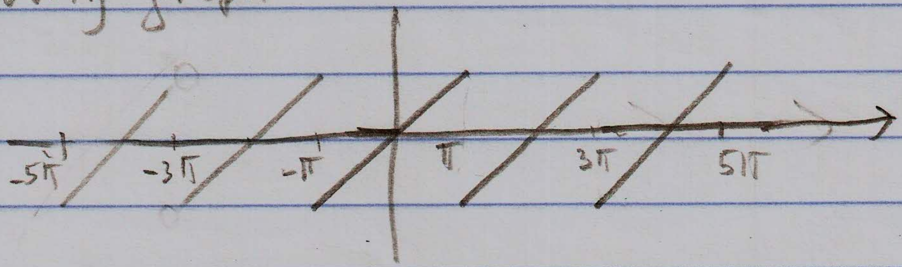
b) Without using Dirichlet Thm (of page 9) to what this Fourier series converge at $x = \pi$?

$$S_N(f)(\pi) = 2\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \underbrace{\sin n\pi}_{0} = 0 \xrightarrow[N\to\infty]{} 0$$

$\Rightarrow S_N(f)(\pi) \to 0$

(c) Using Dirichlet's Thm, sketch the graph of the function the Fourier series converge to?

The periodic function with period 2π such that $f(x) = x$ for $-\pi < x < \pi$ has the following graph



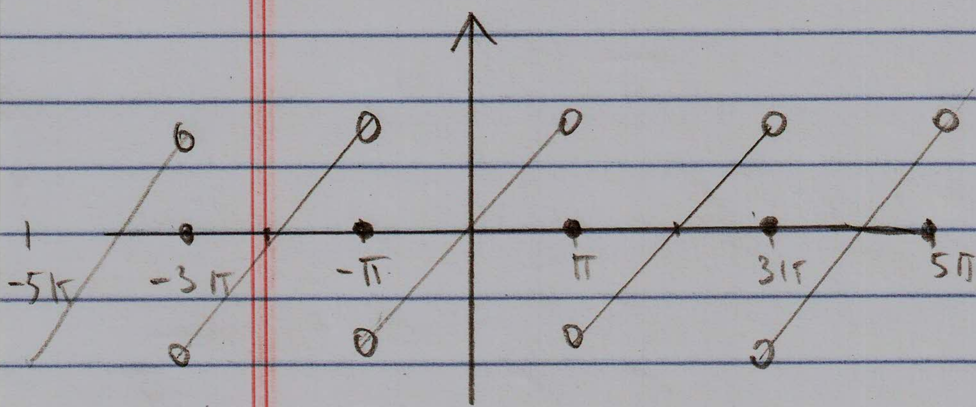
This function has discontinuity at $x = \pi(2l+1)$

and for such x $\quad \frac{f(x-)+f(x+)}{2} = \frac{\pi-\pi}{2} = 0 \Rightarrow$

The Fourier series converge to the function

$$\tilde{f}(x) = \begin{cases} x - 2\pi l, & \text{if } \pi(2l-1) < x < \pi(2l+1) \\ 0 & \text{if } x = \pi(2l+1) \end{cases}$$

The graph of the function $\tilde{f}$ is



(d) Use Parseval's identity for the Fourier series coefficients to obtain the following remarkable formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Parceval's identidy in our case have the form

$$\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx}_{\frac{2\pi^3}{3}} = \sum_{n=1}^{\infty} b_n^2 \qquad \text{where } b_n = \frac{(-1)^{n+1}2}{n} \Rightarrow$$

$$b_n^2 = \frac{4}{n^2}$$

(note that $a_n = 0$)

$$\frac{1}{\pi}\cdot\frac{2\pi^3}{3}$$

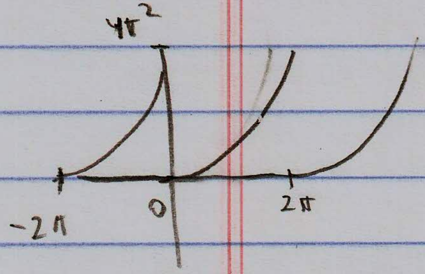
$$\Downarrow$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}} \qquad \text{q.e.d}$$

Next I give an additional example (I did not discuss it in class but it is worth to review it)

Example 2

Expand $f(x)=x^2$, $0<x<2\pi$ in Fourier series of



Solution

As was mentioned at the end of page 7 - beginning of page 8 in order to calculate the coefficients of the Fourier series of the period $2L$ we can take the corresponding integrals over any interval of the length $2L$ (see formulas (10) & (11) on page 8)

In our case $2L=2\pi \Rightarrow L=\pi$ and we can use the following:

$$a_n=\frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} x^2dx=\frac{8\pi^3}{3}$$

$$\underset{n\geq 1}{a_n}=\frac{1}{\pi}\int_0^{2\pi} x^2\cos nx \underset{\substack{\text{integration}\\ \text{by parts}}}{=} \underbrace{\frac{1}{\pi}\frac{x^2\sin nx}{n}\Big|_0^{2\pi}}_{0}-\frac{2}{\pi n}\int_0^{2\pi} x\sin nx\,dx=$$

$$=\frac{2}{\pi n^2}\underbrace{x\cos nx\Big|_0^{2\pi}}_{2\pi}-\frac{2}{\pi n^2}\underbrace{\int_0^{2\pi}\cos nx\,dx}_{0}=\frac{4}{n^2} \Rightarrow a_n=\frac{4}{n^2},\ n\geq 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx = -\frac{1}{\pi}\, x^2 \frac{\cos nx}{n}\Big|_0^{2\pi} + \frac{2}{\pi n}\int_0^{2\pi} x \cos nx\,dx$$

$$= -\frac{1}{\pi}\frac{4\pi^2}{n} + \underbrace{\frac{2}{\pi n^2}\, x \sin nx\Big|_0^{2\pi}}_{0} - \frac{2}{\pi n^2}\underbrace{\int_0^{2\pi} \sin nx\,dx}_{0}$$

$= -\frac{4\pi}{n}$ $\Rightarrow$ The Fourier series is

$$\boxed{\underbrace{\frac{4\pi^2}{3}}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)}$$

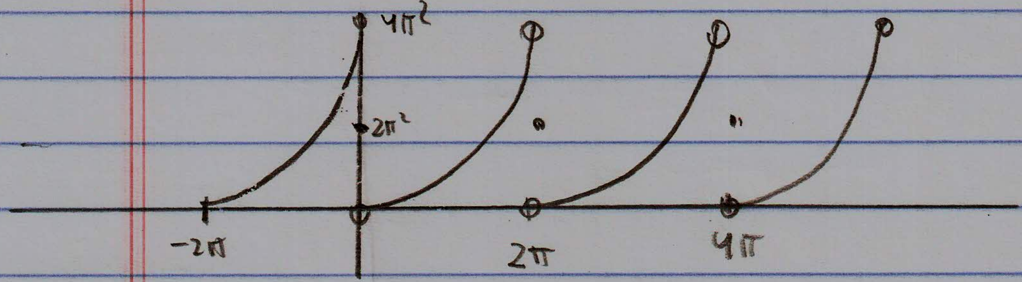
b) Sketch the graph of the function the Fourier series converge to.

Solution If we extend f to the whole $\mathbb{R}$ as a function with period 2π, then the points of discontinuity of this functions are $x = 2\pi n$ and

$$\frac{1}{2}\Big(\underbrace{f(x_-)}_{4\pi^2} + \underbrace{f(x_+)}_{0}\Big) = 2\pi^2 \Rightarrow \text{the Fourier}$$

series converges to $\tilde{f} = \begin{cases} (x - 2\pi l)^2 & 2\pi l < x < 2\pi(l+1) \\ 2\pi^2 & x = 2\pi l \end{cases}$

The graph of $\tilde{f}$ is



(c) Prove the identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

using the expansion of item (a).

<u>Solution</u> Let $x = 2\pi$ then plugging this into

$$\frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

we get $\frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$ $\left(\begin{array}{c}\text{we use here that} \\ \cos 2\pi n = 1\end{array}\right)$
$\sin 2\pi n = 0$

On the other hand it should be equal to $\tilde{f}(2\pi) = 2\pi^2$

$$\Downarrow$$

$$2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2} - \frac{\pi^2}{3} = \frac{\pi^2}{6}$$