MATH 309 PDE part

Lecture 5 April 19

One more example of calculation of Fourier series (was not given in class)

Example 1

Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & -L < x < 0 \\ c & 0 < x < L \end{cases}$$

of the period $2L$ (where c is a constant)

$$a_0 = \frac{1}{L}\int_0^L c\,dx = c$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}\,dx = \frac{1}{L}\int_0^L c\cos\frac{\pi n x}{L}\,dx =$$

because $f(x)=0$ for $-L<x<0$

$$= \frac{c}{L}\,\frac{\sin\frac{\pi n x}{L}}{\frac{\pi n}{L}}\Big|_0^L = 0 \qquad n>0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}\,dx = \frac{1}{L}\int_0^L c\sin\frac{\pi n x}{L}\,dx =$$

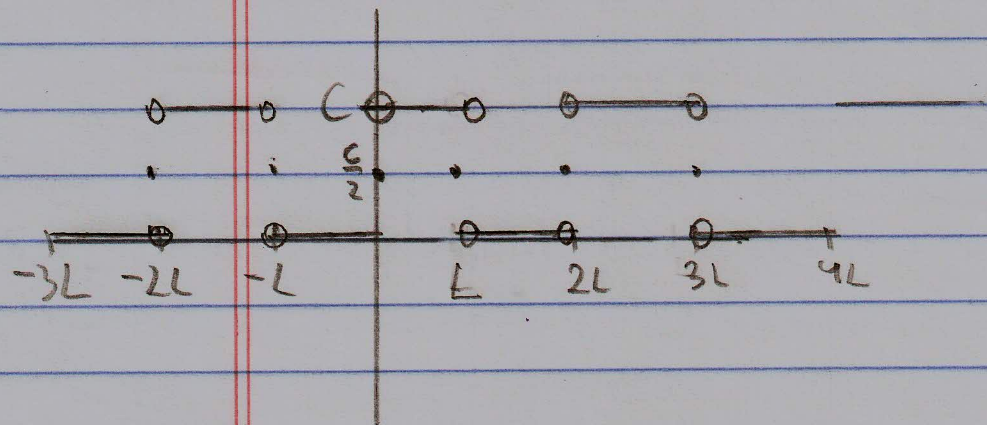
$$= -\frac{1}{L}\,\frac{\cos\frac{\pi n x}{L}}{\frac{\pi n}{L}}\Big|_0^L = -\frac{1}{\pi n}(\cos\pi n - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2c}{\pi n} & n \text{ is odd} \end{cases}$$

$\Rightarrow$ The Fourier series is

$$\frac{c}{2} + \sum_{\ell=1}^{\infty} \frac{2\pi c}{\pi(2\ell-1)} \cos\left(\frac{\pi(2\ell-1)}{L}x\right)$$

(here I plugged $n=2\ell-1$, because odd n only give a contribution)

To what function the Fourier series converges to (sketch also the graph)



$$\frac{c}{2} + \sum_{l=1}^{\infty} \frac{2c}{\pi(2l-1)} \cos \frac{\pi(2l-1)}{L} x = \begin{cases} c, & 2nL < x < (2n+1)L \\ 0, & (2n+1)L < x < 2(n+1)L \\ \frac{c}{2}, & x = Ln \end{cases}$$

c) Using Parseval's identity prove that

$$\sum_{l=1}^{\infty} \frac{1}{(2l-1)^2} = \frac{\pi^2}{8}$$

Consider the case $\underline{c=1}$

Parseval's identity is

$$\|f\|^2 = \frac{a_0^2}{2} + \sum_{l=1}^{\infty} b_{2l-1}^2 = \frac{1}{2} + \sum \frac{4}{\pi^2(2l-1)^2}$$

$$\|f\|^2 = \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{1}{L}\int_0^L dx = 1 \Rightarrow$$

$$1 = \frac{1}{2} + \sum \frac{4}{\pi^2(2l-1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{l=1}^{\infty} \frac{1}{(2l-1)^2}$$

Half-range Fourier series

As we already have seen (from Example of lecture 3 p. 1 - 4 there and homework # 10, problem 8) sometimes we have to expand a function f, defined on an interval $[0, L]$ into the series of sines only or cosines only.

To explain how to do this, let us recall the notion of odd and even functions.

The function f on the interval $[-a, a]$ is called

<u>odd</u> if $\boxed{f(-x) = -f(x)}$ for all $x \in [-a, a]$

and <u>even</u> if $f(-x) = f(x)$ for all $x \in [-a, a]$

<u>Examples</u> • $f(x) = x^n$ is odd if n is odd and even if n is even

Indeed $f(-x) = (-x)^n = (-1)^n x^n = (-1)^n f(x) =$

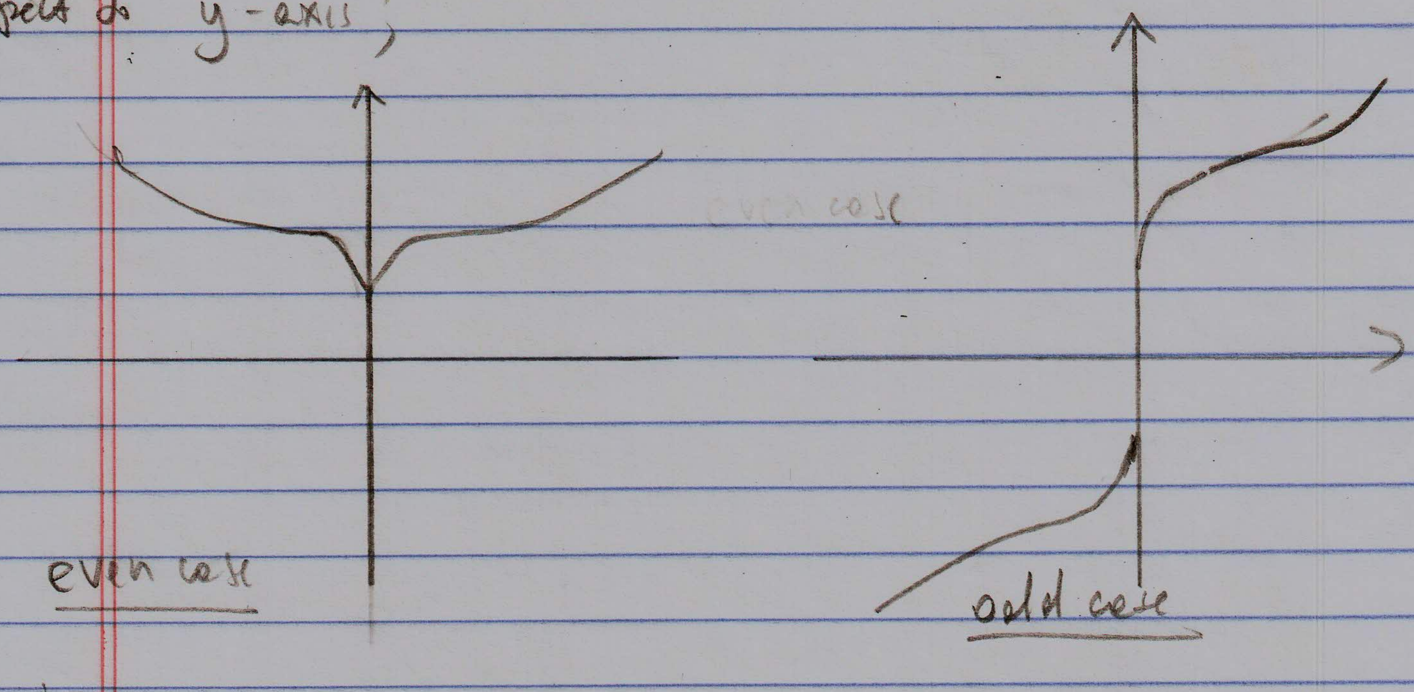
$$= \begin{cases} -f(x), & \text{if } n \text{ is odd} \\ f(x), & \text{if } n \text{ is even} \end{cases}$$

(4)

- $f(x) = \sin \frac{\pi n x}{L}$ is odd,

  $f(x) = \cos \frac{\pi n x}{L}$ is even.

Geometrically the graph of an even function is obtained from its part corresponding to $x > 0$ by reflecting with respect to y-axis;



The graph of an odd function is obtained from its part corresponding to $x > 0$ by central symmetry with respect to the origin $((x,y) \to (-x,-y))$ or, equivalently, by 2 reflections: first with respect to y-axis and second with respect to x-axis (or vice versa).

Main properties of odd and even function

- If f is odd then $\int_{-a}^{a} f(x)\,dx = 0$

- If f is even then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

- Table of the multiplication:

f	g	fg
even	even $\Rightarrow$	even
even	odd $\Rightarrow$	odd
odd	even $\Rightarrow$	odd
odd	odd $\Rightarrow$	even

Note that this table is the same as the table of multiplication of numbers 1 and -1 if we assign -1 to odd functions and 1 to even functions

Fourier series of odd and even functions

Assume that a function f is odd on $(-L, L)$

then $\underbrace{f(x)}_{\text{odd}}\ \underbrace{\cos\frac{\pi n x}{L}}_{\text{even}}$ is odd $\Rightarrow \int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}\,dx = 0 \Rightarrow$

the coefficients a_n of its Fourier series of period $2L$ $\boxed{\text{vanish}}$,

i.e. $\boxed{a_n = 0}$ $n \geq 0$

$\boxed{a_n = 0}$ $\quad n \geq 0$

Therefore the Fourier series (of period 2L) of an odd function has the form

$$\sum b_n \sin \frac{n\pi x}{L}$$

Moreover, $b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\sin\frac{\pi n x}{L}}_{\text{odd}}}_{\text{even}}\, dx = \boxed{\frac{2}{L}\int_{0}^{L} f(x)\sin\frac{\pi n x}{L}\,dx}$

Now assume that a function f is even on $(-L,L)$,

then $\underbrace{f(x)}_{\text{even}}\,\underbrace{\sin\frac{\pi n x}{L}}_{\text{odd}}$, $n \geq 1$ is odd $\Rightarrow \int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}\,dx = 0$

$\Rightarrow$ the coefficients b_n of its Fourier series of period 2L

$\boxed{\text{vanishes}}$, i.e. $b_n = 0$ $\quad n \geq 1 \Rightarrow$

The Fourier series (of period 2L) of an even function has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

Moreover

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\cos\frac{\pi n x}{L}}_{\text{even}}}_{\text{even}} dx = \frac{2}{L}\int_0^L f(x)\cos\frac{\pi n x}{L}\, dx$$

All this motivates the following definition

Def Let f be a (Riemann integrable) function defined on the interval $(0,L)$. A series

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx \qquad (1)$$

is called the (half-range) Fourier sine series of f on $(0,L)$. A series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx \qquad (2)$$

is called the (half-range) Fourier cosine series of f on $(0,L)$.