In other words, to obtain a Fourier sine series of f is the same as to extend the function f from $(0,L)$ to $(-L,L)$ in the odd way, i.e. such that $f(-x)=-f(x)$ and then to find the Fourier series (of period $2L$) of the obtained function.

Similarly, to obtain a Fourier cosine series of f is the same as to extend the function f from $(0,L)$ to $(-L,L)$ in the even way, i.e such that $f(-x)=f(x)$ and to find the Fourier series (of period $2L$) of the obtained function.

Example 2 a) Expand $f(x)=\sin x$, $0<x<\pi$ in the Fourier cosine series

(Preliminary question: What is the Fourier sine series of this f? Answer: $\sin x$ itself

In general if f is of the form $\sum_{n=1}^{N} B_n \sin\frac{n\pi x}{L}$, then its Fourier sine series is $\sum_{n=1}^{N} b_n \sin \frac{n\pi x}{L}$)

Solution $L=\pi \Rightarrow \frac{n\pi x}{L} = nx$

By formula (2)

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{\sin x}_{f(x)} \cos nx \, dx = (*)$$

Use the trigonometric identity

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$$

Putting $\alpha = x$ and $\beta = nx$

$$(*) = \frac{\cancel{2}}{\pi}\int_0^{\pi}\frac{1}{\cancel{2}}\left(\sin\underbrace{(x+nx)}_{(n+1)x} + \sin\left(\underbrace{x-nx}_{(1-n)x}\right) dx\right) dx =$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin((n+1)x) - \sin((n-1)x)\,dx \underset{(\text{if } n\neq 1)}{=} \frac{1}{\pi}\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\Big|_0^{\pi}\right) =$$

$$= \frac{1}{\pi}\left(\frac{\cos(n-1)\pi - 1}{n-1} - \frac{\cos(n+1)\pi - 1}{n+1}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{1}{\pi}\left(\frac{2}{n+1} - \frac{2}{n-1}\right) & \text{if } n \text{ is even}\end{cases}$$

Note that

$$\frac{1}{\pi}\left(\frac{2}{n+1} - \frac{2}{n-1}\right) = \frac{1}{\pi}\,\frac{2(n-1)-2(n+1)}{(n+1)(n-1)} = -\frac{4}{\pi(n^2-1)}$$

Note that $a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx = 0$

So

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{\pi(n^2-1)} & \text{if } n \text{ is even}\end{cases}$$

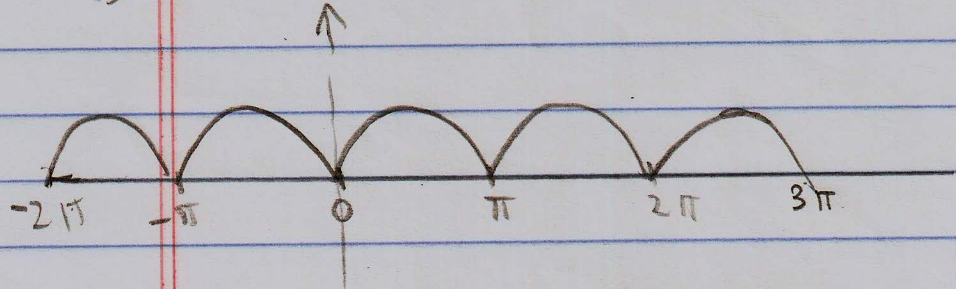
$\Rightarrow$ the Fourier cosine series is

$$\boxed{\frac{2}{\pi} - \frac{4}{\pi}\sum_{\ell=1}^{\infty} \frac{4}{\pi(4\ell^2-1)} \cos(2\ell x)}$$

(in the last formula I plug $n=2\ell$ Because only even n give a contribution to the Fourier cosine series)

b) Which is the function the Fourier cosine series converge to? Sketch the graph

The Fourier cosine series converges to a <u>even</u> function of period $2L = 2\pi$. To get this function we first have to extend sinx in the even manner from $(0,\pi)$ to $(-\pi,\pi)$ and then to extend the obtained function to the whole $\mathbb{R}$ as a periodic function of period 2π The graph is



Actually the required function is $\boxed{|\sin x|}$

(note that this function is continuous, ~~with piecewise~~ cont-

-11-

nuous derivative. Therefore the Fourier cosine series indeed converge to it (by Dirichlet's Theorem)

Example 3 (it was not given in class)

Expand $f(x) = x$, $0 < x < \pi$ in half range series

a) Fourier sine series. Sketch what the series converge to

Extending $f(x) = x$ from the interval $(0, \pi)$ to the interval $(-\pi, \pi)$ in the odd manner, we will get

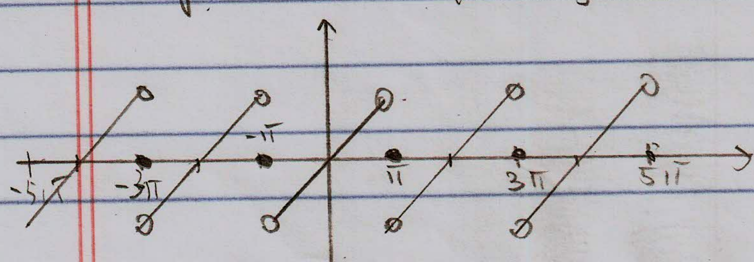
$$f(x) = x \quad , -\pi < x < \pi, \text{ i.e.}$$

exactly the function considered in Example 1

p 10-13 of lecture 4 of class 04/17. So the answer here is the same as in that example

i.e., $= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

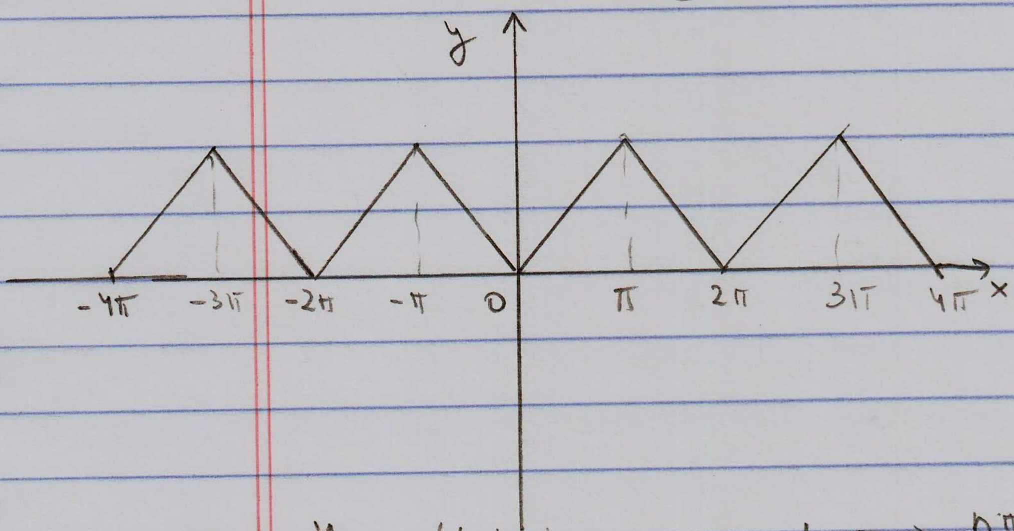
The series converge to the function obtained from $f(x) = x$ $-\pi < x < \pi$ by extending it as a periodic odd function with period 2π (taking also the midpoints of the jumps)



6) Fourier cosine series. Sketch what the series converge to

This time we have to extend $f(x)=x$ from $(0,\pi)$ to $(-\pi,\pi)$ in the <u>even</u> manner and then to the whole $\mathbb{R}$ as a periodic function with a period 2π

We get the following graph



Find the coefficients a_n: $\quad L=\pi \Rightarrow \frac{n\pi x}{L}=nx$

For $n\geq 1$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\underbrace{x}_{f(x)}\cos nx\,dx \underset{\substack{\text{integration}\\ \text{by parts}}}{=} \underbrace{\frac{2}{\pi}\,\frac{x\sin nx}{n}\Big|_0^{\pi}}_{=0}-\frac{2}{\pi n}\int_0^{\pi}\sin nx\,dx=$$

$$=\frac{2}{\pi n^2}\cos nx\Big|_0^{\pi}=\frac{2}{\pi n^2}(\cos\pi n-1)=\begin{cases}0, & \text{if } n \text{ is even}\\ -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd}\end{cases}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\cdot\frac{\pi^2}{2}=\pi \Rightarrow$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\boxed{\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\ldots\right)}$$

$$=\boxed{\frac{\pi}{2}-\frac{4}{\pi}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell-1)^2}\cos(2\ell-1)x}$$