MATH 309 PDE part Igor Zelenko

Lecture 6 April 22

Applications to the heat equation

Consider the heat equation

(1) $u_t = k u_{xx}$ on $0<x<L$, $t>0$

with initial condition (2) $u(x,0)=\varphi(x)$, where $\varphi(x)$ is piecewise continuous and has piecewise continuous derivative on $[0,L]$

We will consider several different boundary conditions separately

Case 1) Boundary conditions $\boxed{u(0,t)=u(L,t)=0}$, $t>0$ (3)
(the ends are kept with zero temperature)

As we already have seen in Lecture 3 (class 04/15) Example 1 pp. 1-3 using the method of separation of variables one arrives to 2 ODE's

$$X''=CX \qquad X(0)=X(L)=0 \qquad (4)$$
$$T'=kCT \qquad (5)$$

Moreover, from the discussions of the same lecture 3 on pages 2 and 3 it follows that for the boundary cond. (4)

where $C = -\frac{\pi^2 n^2}{L^2}$ where n is a natural number

and any solution $u(x,t)$ of the type $X(x)T(t)$ has the form

$$u(x,t) = B_n e^{-\frac{k\pi^2 n^2}{L^2}t} \sin\frac{\pi n}{L}x \qquad (5)$$

Now we are looking for the solution of (1) which satisfies also initial conditions (2) in the form of an (infinite in general) superposition (i.e. linear combination) of solutions of type (5)

i.e. in the form

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{k\pi^2 n^2}{L^2}t} \sin\frac{\pi n}{L}x \qquad (6)}$$

We want that $u(x,0) = \varphi(x)$, i.e.

$$\varphi(x) = \sum_{n=1}^{\infty} B_n \sin\frac{\pi n}{L}x$$

, i.e. the right hand side has to be the Fourier sine series of φ and

$$B_n = \frac{2}{L}\int_0^L \varphi(x)\sin\left(\frac{\pi n}{L}x\right)dx$$

So, at least formally, the solution of our initial/boundary value problem has to be of the form

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\left(\int_0^L \varphi(\xi)\sin\frac{\pi n \xi}{L}d\xi\right) e^{-k\frac{\pi^2 n^2}{L^2}t} \sin\frac{\pi n}{L}x \qquad (7)}$$

Remark 1 As a matter of fact, one needs to justify that $u(x,t)$ is indeed the solution of the initial/boundary value problem on $0<x<L$, $t>0$. For example, one needs to justify that it is eligible to differentiate the infinite series (6) term by term twice with respect to x and once with respect to t. This can be done using another notion of convergence of the series, the uniform convergence (but we do not have time and enough mathematical background to discuss this here).

Remark 2 Note also that in general we do not assume that $\varphi(0)=0$ and $\varphi(L)=0$ so the initial conditions and the boundary conditions do not match at the corner points $(x,t)=(0,0)$ and $(L,0)$ (this is the reason why we write $t>0$ and not $t\geq 0$ in (3)). Moreover one can ensure the solution given by the formula (6) converge to $\varphi(x)$ when $t\to 0$ only at the points of continuity of φ.

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Remark 3 (smoothing effect) Note that even if φ is only piecewise continuous, the function $u(x,t)$ for $t>0$ has continuous derivative of any order with respect to x and t, i.e. the process of heat flow described by the heat equation is a smoothing process.

To summarize In order to solve the initial/boundary value problem (1)–(3) (of page 1):

1) Find the expansion of φ into the Fourier sine series.

2) Plug the coefficients b_n that you found in step 1 into the formula (6) (of page 2)

Case 2) Boundary conditions

$$\boxed{u_x(0,t) = u_x(L,t) = 0} \quad (8)$$

(insulated ends)

Applying the method of separation of variables, one arrive to the same two ODEs

$$X'' = cX \qquad (4')$$

$$T' = \kappa c T \qquad (5')$$

but with different boundary conditions of for $X(x)$, namely,

$$X'(0) = X'(L) = 0$$

Let us show that if $X(x) \not\equiv 0$ in this case the constant c in (4') must be nonpositive, i.e $c \le 0$

Indeed if $c > 0 \Rightarrow X(x) = Ae^{\sqrt{c}x} + Be^{-\sqrt{c}x} \Rightarrow$

$\Rightarrow X'(x) = \sqrt{c}Ae^{\sqrt{c}x} - \sqrt{c}Be^{-\sqrt{c}x} \Rightarrow$

$$X'(0) = X'(L) = 0 \Rightarrow \begin{cases} \sqrt{c}A - \sqrt{c}B = 0 \\ \sqrt{c}Ae^{L\sqrt{c}} - \sqrt{c}Be^{-\sqrt{c}L} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} A - B = 0 \\ e^{L\sqrt{c}}A - e^{-L\sqrt{c}}B = 0 \end{cases}$$ We get a system of 2 equations for A & B

The determinant of this system is

$$\begin{vmatrix} 1 & -1 \\ e^{L\sqrt{c}} & -e^{-L\sqrt{c}} \end{vmatrix} = -e^{-L\sqrt{c}} + e^{L\sqrt{c}} > 0 \Rightarrow A = B = 0, \text{ but}$$

we are looking for a non-zero solution

So $c \leq 0$

If $c=0 \Rightarrow X''=0 \Rightarrow X(x)=Ax+B$

The boundary conditions imply that

$A \left(= X'(0)\right)$ vanish $\Rightarrow X(x) \equiv \underbrace{B}_{\text{constant}}$

If $c<0$ then $c=-\lambda^2 \Rightarrow$

$$X(x) = A\cos\lambda x + B\sin\lambda x \Rightarrow$$
$$X'(x) = -A\lambda\sin\lambda x + \lambda B\cos\lambda x \Rightarrow$$

$X'(0)=0 \Rightarrow \lambda B = 0 \Rightarrow B=0$

$X'(L)=0 \Rightarrow -A\lambda\sin\lambda L = 0 \Rightarrow \lambda L = \pi n \Rightarrow \lambda = \frac{\pi n}{L}$

where $n \in \mathbb{N}$ (note that n & $-n$ give the same solution up to a constant multiple)

$\Downarrow$

$$X(x) = A\cos\frac{\pi n}{L}x \Rightarrow \begin{cases} c = -\frac{\pi^2 n^2}{L^2} \\ T(t) = e^{-k\frac{\pi^2 n^2}{L^2}t} \end{cases}$$

Hence any solution satisfying eq (1) and boundary conditions (3) and having the form $X(x)T(t)$ is of the form $u(x,t) = a_n e^{-k\frac{\pi^2 n^2}{L^2}t}\cos\frac{\pi n}{L}x \qquad n \geq 0$

($n=0$ corresponds to $u(x,t) \equiv$ const)

By the complete analogy with the previous case we look for the solution of the initial/Boundary problem given by (1), (2), (8) in the form

$$u(x,t) = \frac{a_0}{2} + \sum a_n e^{-k\frac{\pi^2 n^2}{L^2}t} \cos\frac{\pi n}{L}x \tag{9}$$

We want that $u(x,0) = \varphi(x)$, i.e.

$$\varphi(x) = \frac{a_0}{2} + \sum a_n \cos\frac{\pi n}{L}x,$$ i.e. the right hand has to be the Fourier cosine series of φ and

$$a_n = \frac{2}{L}\int_0^L \varphi(x)\cos\left(\frac{\pi n}{L}x\right)dx$$

So, at least formally, the solution of our initial/boundary value problem is of the form

$$u(x,t) = \frac{1}{L}\int_0^L \varphi(\xi)\,d\xi + \sum_{n=1}^{\infty}\frac{2}{L}\left(\int_0^L \varphi(\xi)\cos\frac{\pi n\xi}{L}\,d\xi\right)e^{-k\frac{\pi^2 n^2}{L^2}t}\cos\frac{\pi n}{L}x \tag{10}$$

(8)

To summarize In order to solve the initial / boundary value problem given by (1), (2), (8)

1) Find the expansion of f into the Fourier cosine series;

2) Plug the coefficients a_n that you found in step 2 into the formula (9) (of page 7)

Case 3) (enrichment will not be asked in the tests) Boundary conditions

$$u(0,T) = u_x(L,T) = 0 \qquad (11)$$

(one end is kept with zero temperature and another end is insulated)

Using the method of separation of variables

We arrive to the ODE Boundary value problem:

$$X'' = cX \qquad X(0) = X'(L) = 0.$$

$$T' = kcT$$

Similarly to lecture 3 p. 2-3 or the current lecture page 5 one can show that if $X(x) \neq 0$, then $c < 0$. Assume that $c = -\lambda^2 \Rightarrow$

$$X'' + \lambda^2 X = 0 \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x$$

$$X'(x) = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$$

$$\Rightarrow \quad X(0) = 0 \Rightarrow A = 0$$

$$X'(L) = 0 \Rightarrow \lambda B\cos\lambda L = 0 \Rightarrow$$

$$\lambda L = \frac{\pi}{2} + \pi n \Rightarrow \lambda = \frac{1}{L}\left(\frac{\pi}{2} + \pi n\right) = \frac{\pi}{L}\left(n + \frac{1}{2}\right)$$

(we can take integer $n \geq 0$) $\Rightarrow$ solution of the eq (1) satisfying boundary conditions (11) of the form $X(x)T(t)$ is

$$u(x,t) = C_n e^{-k\frac{\pi^2}{L^2}\left(n+\frac{1}{2}\right)^2 t} \sin\left(n+\frac{1}{2}\right)\frac{\pi}{L}x$$

Then by analogy with the case 1) the solution of the boundary/initial value problem given by (1), (2), (11) is of the form

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{\left(-k\frac{\pi^2}{L^2}\left(n+\frac{1}{2}\right)^2 t\right)} \sin\left(n+\frac{1}{2}\right)\frac{\pi}{L}x, \text{ where}$$

$$c_n = \frac{2}{L}\int_0^L \varphi(x)\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{L}x\right)dx, \quad n \geq 0$$