April 22

Lecture 6 Application to the Laplace equation on the rectangle.

Consider the Laplace equation

(12) $u_{xx}+u_{yy}=0$ on the rectangle $0\le x\le L$
$0\le y\le H$

In (x,y)-plane.

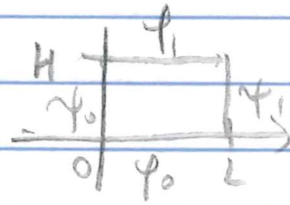


Fig. 1

The Dirichlet boundary condition

$$u(x,0)=\varphi_0(x),\quad 0<x<L$$
$$u(x,H)=\varphi_1(x),\quad 0<x<L$$
$$u(0,y)=\psi_0(y)\quad 0<y<H$$
$$u(L,y)=\psi_1(y)\quad 0<y<H$$

or, shortly, u is prescribed on the boundary of the rectangle (see Fig. 1 that describes this boundary conditions schematically)

To solve this boundary problem we can solve separately 4 boundary value problems given by the following figures

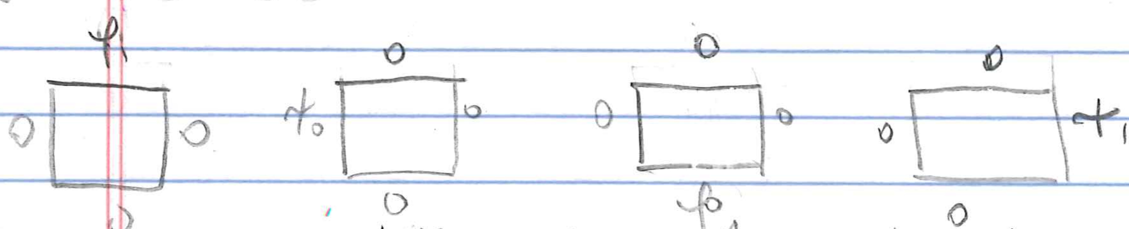
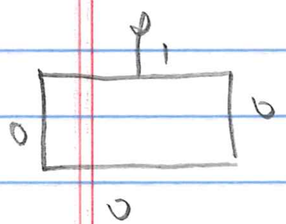


Fig 2

(In other words, in each of these problems the boundary function is zero on 3 out of 4 edges)

Then, by superposition principle, the solution of the original boundary value problem is the sum of the solutions of each subproblems described in Fig. 2.

Let us demonstrate how to solve the boundary value problem given by



, i.e when

$u(x,0)=0$, $u(0,y)=u(L,y)=0$ $\quad 0<y<H$ (13)
$0<x<L$

and $u(x,H)=\varphi_1(x)$, $0<x<L$ (14)

Use the method of separation of variables

$u_{xx}+u_{yy}=0 \qquad u=X(x)Y(y) \Rightarrow$

$X''Y+XY''=0 \Rightarrow \dfrac{X''}{X}=-\dfrac{Y''}{Y}=c \Rightarrow$

$$\begin{cases} X''-cX=0 & (15) \\ Y''+cY=0 & (16) \end{cases}$$

Besides $u(0,y)=u(L,y)=0 \Rightarrow X(0)Y(y)=X(L)Y(y)=0 \Rightarrow$

since we are looking for nonzero solutions (and therefore $Y(y)\neq 0$) we get $X(0)=X(L)=0$

So for X we have the following ODE boundary value problem

$$X'' - cX = 0$$
$$X(0)=X(L)=0$$

We already have discussed it in Lecture 3 page 2,3 (and also in the current lecture on pages 1 & 2)

So if $X(x)\neq 0$ $\quad c = -\frac{\pi^2 n^2}{L^2}$, $n\in N$

$$X(x) = B\sin\frac{n\pi x}{L} \qquad (17)$$

Substituting the value of $c = -\frac{\pi^2 n^2}{L^2}$ into the equation

(16) : $Y'' - \frac{\pi^2 n^2}{L^2} Y = 0 \Rightarrow$

$$Y(y) = Ae^{\frac{\pi n}{L}y} + Be^{-\frac{\pi n}{L}y}$$

Since $u(x,0)=0$, $Y(0)=0 \Rightarrow A+B=0 \Rightarrow$

$$B = A \Rightarrow Y(y) = A\left(e^{\frac{\pi n}{L}y} - e^{-\frac{\pi n}{L}y}\right) =$$

$$= 2A\underbrace{\sinh}_{\text{the hyperbolic sine.}}\left(\frac{\pi n}{L}y\right) \qquad (17)$$

About hyperbolic trig. functions:

$$(17)\begin{cases} \cosh y := \dfrac{e^{y}+e^{-y}}{2} \to \text{hyperbolic cosine} \\ \sinh y := \dfrac{e^{y}-e^{-y}}{2} \to \text{hyperbolic sine} \end{cases}$$

Properties: $\cosh^2 y - \sinh^2 y = 1$ ("−" instead of "+" compared to the usual trig. identity)

· $(\cosh y)' = \sinh y$, $(\sinh y)' = \cosh y$

(Recall that usual sine and cosine also can be expressed similarly to (16) by

$$\cos y = \frac{e^{iy}+e^{-iy}}{2}, \quad \sin y = \frac{e^{iy}-e^{-iy}}{2i}$$

via the Euler identity)

Returning to our boundary value problem:

From (16) and (17) we get that the solutions of the form $X(x)Y(y)$ satisfying the equation (12) and the boundary conditions (13) are

$$u(x,y) = C_n \sinh\left(\frac{\pi n}{L} y\right) \sin \frac{\pi n}{L} x \Rightarrow$$
$$n \in \mathbb{N}$$

If we also want that the boundary condition (14), i.e. $u(x,H)=\varphi_1(x)$ will hold we are looking

for the solution in the form

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{\pi n}{L} y\right) \sin\left(\frac{\pi n}{L} x\right) \qquad (18)$$

and we want that $u(x,H) = \sum C_n \sinh\left(\frac{\pi n}{L} H\right) \sin\left(\frac{\pi n}{L} x\right) = \varphi_1(x)$

Therefore, if $\sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n}{L} x\right)$ is the Fourier sine

series of φ_1 on $(0,L)$ then $B_n = \frac{2}{L}\int_0^L \varphi_1(\xi) \sin\frac{n\pi\xi}{L}\, d\xi$

and

$$C_n \sinh\left(\frac{\pi n}{L} H\right) = B_n \Rightarrow C_n = \frac{B_n}{\sinh\left(\frac{\pi n}{L} H\right)} \qquad (19)$$

or, equiv., $C_n = \dfrac{2}{L \sinh\left(\frac{\pi n}{L} H\right)} \displaystyle\int_0^L \varphi_1(\xi) \sin\frac{n\pi\xi}{L}\, d\xi \qquad (19')$.

So the required solution is (18) with C_n satisfying (19)

i.e.

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{L \sinh\left(\frac{\pi n}{L} H\right)} \left(\int_0^L \varphi_1(\xi) \sin\frac{\pi n \xi}{L}\, d\xi\right) \sinh\frac{\pi n}{L} y \sin\frac{\pi n}{L} x$$

To summarize, in order to solve the boundary value problem (12)-(14) we can

1) Expand $\varphi_1(x)$ in the Fourier sine series

2) Use formula (18) with c_n satisfying (19) with the coefficients b_n found in step 1 (or, equivalently, with c_n satisfying (19'))

Other 3 boundary problem, schematically described by Fig. 2 of page 2 can be treated similarly

One can give similar application to the wave equation (but we have no time to do it)