Lecture 7 April 26 MATH 322

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Summarizing table for separation in cartesian coordinates

	Heat equation		Laplace eq	Wave equation (was not discussed in detail)
	$u_t = k u_{xx}$		$u_{xx} + u_{yy} = 0$	$u_{tt} - a^2 u_{xx} = 0$
ODE's obtained by separation	$X'' = cX$, $T' = cT$		$X'' = cX$, $Y'' = -cY$	$X'' = cX$, $T'' = a^2 c T$
Boundary cond.	$u(0,t) = u(L,t)$ ⇓ $X(0) = X(L) = 0$	$u_x(0,t) = u_x(0,L)$ ⇓ $X'(0) = X'(L)$	$u(0,y) = u(L,y)$ ⇓ $X(0) = X(L) = 0$	$u(0,t) = u(L,t)$ ⇓ $X(0) = X(L) = 0$
Possible values of c (spectrum)	$-\frac{\pi^2 n^2}{L^2}$	$-\frac{\pi^2 n^2}{L^2}$	$-\frac{\pi^2 n^2}{L^2}$ If in addition $u(x,0) = 0$	$-\frac{\pi^2 n^2}{L^2}$ If in addition $u_t(x,0) = 0$
Solution of type $X(x)T(t)$ or $X(x)Y(y)$	$Ce^{-\frac{\pi^2 n^2}{L^2}t} \sin\frac{\pi n}{L}x$	$Ce^{-\frac{\pi^2 n^2}{L^2}t} \cos\frac{\pi n}{L}x$	$C \sinh\frac{\pi n}{L}y \sin\frac{\pi n}{L}x$	$C \cos\frac{\pi n}{L}at \sin\frac{\pi n}{L}x$

And then we have one more boundary or initial condition, and we look for the solution as an infinite linear combination of the solutions of the bottom column.

PDE part MATH 309 Igor Zelenko

Lecture 7 Separation of variable for 04/24

equation

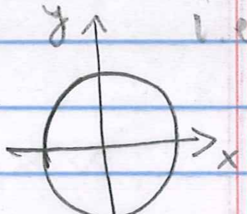
PDE on circular and cylindrical regions

Application 1 Laplace equation on a disk

Consider the Laplace equation

(1) $$u_{xx}+u_{yy}=0 \quad \text{on the unit disk } x^2+y^2<1$$

with the Dirichlet boundary condition
i.e. such that u is prescribed on the boundary, which the unit circle here



It is natural here to pass to the polar coordinates: $x=r\cos\theta$
$y=r\sin\theta$

Then using the chain rule one can show that

$$u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}$$

So the equation (1) in the new coordinates has the form

$$u_{rr}+\frac{1}{r}u_r+u_{\theta\theta}=0 \quad (2)$$

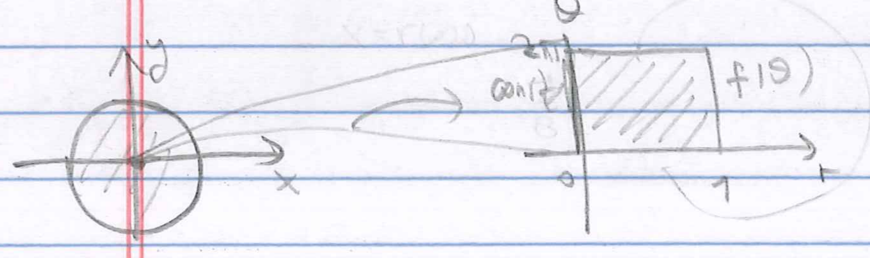
and the Dirichlet boundary condition are given by

$$u(\underbrace{1}_{\substack{\text{since we}\\ \text{are on the}\\ \text{unit circle}}},\theta)=f(\theta),$$ where

$f(\theta)$ is a periodic function of period 2π

(say piecewise continuous with piecewise continuous derivative)

Although eq (2) looks more complicated the passage to polar coordinates transforms the disk to the rectangle



and the boundary conditions are:

$u(1,\theta) = f(\theta)$

$u(0,\theta) = \text{const}$ (the origin goes to the left edge)

$u(r,\theta)$ is periodic of period 2π

Let us separate the polar variables, i.e. look for the solution in the form $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\Rightarrow \quad R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\,\Theta'' = 0$$

$$\Updownarrow$$

$$\frac{r^2\left(R'' + \frac{1}{r}R'\right)}{R} = -\frac{\Theta''}{\Theta} = c \Rightarrow$$

$$r^2R'' + rR' - cR = 0 \qquad (3)$$ → the Euler equation (chapter 5 of the textbook of MA0H30s)

$$\Theta'' + c\Theta = 0 \qquad (4)$$

$\Theta(\theta)$ is periodic of period $2\pi \Rightarrow c \geq 0$

Either Θ is constant or $\Theta = A\cos k\theta + B\sin k\theta$

$\Rightarrow$ either Θ is constant $\Rightarrow \lambda = 0$ or $\Theta(\theta) = A\cos\lambda\theta + B\sin\lambda\theta$ with $\lambda > 0$.

the function $A\cos\lambda\theta + B\sin\lambda\theta$ is periodic with all possible periods $\frac{2\pi}{\lambda}n$, $n \in \mathbb{N} \Rightarrow \frac{2\pi}{\lambda}n = 2\pi$ for some $n \Rightarrow \lambda$ must be natural number $\Rightarrow$

$\Theta(\theta) = a_n \cos n\theta + b_n \sin n\theta$

$c = n^2 \Rightarrow$ substitute to Eq (3)

$$r^2 R'' + r R' - n^2 R = 0 \qquad (5) \qquad \text{Euler equation}$$

Recall that we are looking for solution in the form

$R(r) = r^\alpha \Rightarrow$ substituting to (5) we get

$$\alpha(\alpha-1) + \alpha - n^2 = 0 \Leftrightarrow \alpha^2 - n^2 = 0 \Rightarrow$$

Case 1 If $n \neq 0$ $\alpha = \pm n$ i.e the general solution of (5)

is $R(r) = Cr^n + Dr^{-n}$ ($\to \infty$ as $r \to 0$) $\Rightarrow$ since $R(0)$

is finite then $D = 0$, i.e $R(r) = Dr^n$

Case 2 If $n = 0$ (repeated root 0) $\Rightarrow$ the solution is

$C + D\ln r$ ($\to -\infty$ as $r \to 0+$) $\Rightarrow D = 0$

To summarize the solution $u(r,\theta) = R(r)\Theta(\theta)$

of (2) such that $\Theta(\theta)$ is periodic with period 2π and $R(0)$ is finite is of the form

$$u(r,\theta) = r^n(a_n \cos n\theta + b_n \sin n\theta), \quad n \in N \cup \{0\}$$

(if $n=0$ then $u(r,\theta) = const$)

By analogy with the previously treated cases

we look for the solutions in the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \qquad (6)$$

~~s.t $u(1,\theta) = f(\theta)$ or equivalently~~

$$\cancel{f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)}$$

~~Another words~~ a_n and b_n are exactly the coefficients

of the Fourier series of $f(\theta)$ (of period 2π) i.e

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n\varphi \, d\varphi$$

$$\cancel{b_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin n\varphi \, d\varphi}$$

Enrichment

As a matter of fact substituting a_n and b_n to (6), regrouping the terms and making some simplification one can get the Poisson's integral formula

$$u(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(\varphi)\, d\varphi}{1+r^2-2r\cos(\theta-\varphi)}$$

Advantage: Explicit formula that does not contain series