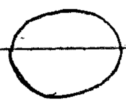
Lecture 8/9 Classes of April 29/30

Bessel's differential equation

Bessel's equation is ODE which appears naturally in separation of variables in cylindrical coordinates

Wave	Heat	Laplace
$u_{tt} = a^2(u_{xx}+u_{yy})$	$u_t = k(u_{xx}+u_{yy})$	$u_{xx}+u_{yy}+u_{zz}=0$
$(x,y) \in$ unit disk	$(x,y) \in$ unit disk	$(x,y,z) \in$ a cylinder $x^2+y^2<1$, $0<z<H$
 → drum	a circle plate (the faces are insulated so that this is a problem in the plane)	

Change to cylindrical coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $t = t$	$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Use that $u_{xx}+u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

$u(x,y,t) = R(r)\Theta(\theta)T(t)$ $\qquad$ $u(x,y,z) = R(r)\Theta(\theta)Z(z)$

The result of separation

$$\frac{T''}{a^2T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -c \qquad \underbrace{\frac{T'}{kT} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}= -c \qquad -\frac{Z''}{Z} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2} = -c$$

$$T'' = -a^2cT \qquad T' = -kcT \qquad Z'' = cZ$$

Seperate further R with Θ

$$\frac{r^2R'' + rR' + cr^2R}{R} = -\frac{\Theta''}{\Theta} = d$$

We want that $\Theta(\theta)$ will be periodic with period 2π ⇒

$\Theta'' + d\Theta = 0$ ⇒ exactly as in the case of Laplace equation in the disk: $d = n^2$ ⇒ we get the following equation for R:

$$\boxed{r^2R'' + rR' + (cr^2 - n^2)R = 0}$$

Now assume that $c>0$ (it will be justified later on p. 13)

and that $c=\lambda^2$.

Then we get the equation

$$r^2R''+rR'+(\lambda^2r^2-n^2)R=0 \qquad (1)$$

Now consider the equation (1) with $\lambda=1$

$$r^2R''+rR'+(r^2-n^2)R=0 \qquad (2)$$

<u>Proposition 1</u> If $R(r)$ is a solution of (2) then $R(\lambda r)$ is a solution of one

<u>Proof</u> Let $\tilde{r}=\lambda r$. Substitute $R(\lambda r)$ into (1) taking into account that $\frac{d}{dr}R(\lambda r)=\frac{d}{d\tilde{r}}R(\tilde{r})\cdot\lambda$, $\frac{d^2}{dr^2}R(\lambda r)=\frac{d^2}{d\tilde{r}^2}R(\tilde{r})\cdot\lambda^2 \Rightarrow$

$$r^2\frac{d^2}{dr^2}R(\lambda r)+r\frac{d}{dr}R(\lambda r)+(\lambda^2r^2-n^2)R=$$

$$\underbrace{\lambda^2r^2}_{\tilde{r}^2}\frac{d^2}{d\tilde{r}^2}R(\tilde{r})+\underbrace{\lambda r}_{\tilde{r}}\frac{d}{d\tilde{r}}R(\tilde{r})+(\underbrace{\lambda^2r^2}_{\tilde{r}^2}-n^2)R(\lambda r)=$$

$$=\tilde{r}^2\frac{d^2}{d\tilde{r}^2}R(\tilde{r})+\tilde{r}\frac{d}{d\tilde{r}}R(\tilde{r})+(\tilde{r}^2-n^2)R(\tilde{r})=0$$

(since $R(r)$ is a solution of (2)) $\Rightarrow$ $R(\lambda r)$ is indeed a solution of (1)

Def The differential equation

$$r^2R'' + rR' + (r^2 - \nu^2)R = 0 \qquad (3)$$

(here ν is not necessarily integer as in (2))

is called the Bessel equation of order n

Sketch of the theory of series solutions of ODE
(Chapter 5 of the textbook of MATH 308)

Given a differential equation

$$P(r)R'' + Q(r)R' + S(r)R = 0 \qquad (4)$$

with P, Q, S being analytic at 0 ($\Leftrightarrow$ can be represented as a power series around 0)

- If $P(0) \neq 0$ (in this case $r=0$ is called a regular point) then solutions of (4) can be found as power series $\sum_{n=0}^{\infty} a_n r^n$

- If $P(0) = 0$ and $\frac{Q(r)}{P(r)}, \frac{S(r)}{P(r)}$ are not finite as $r \to 0$ (our case) then $r=0$ is called singular. If in addition

$$P(r) = ar^2 + \text{higher order terms}, \quad a \neq 0$$
$$Q(r) = br + \text{higher order terms} \qquad (5)$$
$$S(r) = c + \text{higher order term}$$

then $r=0$ is called regular singular and at least one solution can be found in the form

$$R(r) = r^{\alpha} \underbrace{(c_0 + c_1 r + c_2 r^2 + \dots)}_{\text{power series}} \qquad (6)$$

(with P, Q R as in (5))

Substituting (5) into (4) it is easy to see that the power α must satisfy the indicial equation

$$\alpha(\alpha-1)a + \alpha b + c = 0 \qquad (7)$$

In other words, r^{α} is the solution of the Euler equation

$$ar^2R'' + brR' + cR = 0$$

obtained from (4) by removing the higher order terms in (6).

More precisely, if α_1 and α_2 are two solution of the quadratic equation (7) such that $Re\,\alpha_1 \geq Re\,\alpha_2$ then at least one solution of (5) has the form

$$R_1(r) = r^{\alpha_1}\left(c_0 + c_1 r + c_2 r^2 + \dots\right) \quad (8)$$

and coefficients $c_0, c_1, c_2, \dots$ can be found recursively by substituting $R_1(r)$ into eq (4)

Moreover, • if $\alpha_1 - \alpha_2$ is not integer then the second independent solution can be found in the form

$$R_2(r) = r^{\alpha_2}\left(\tilde{c}_0 + \tilde{c}_1 r + \tilde{c}_2 r^2 + \dots\right) \quad (9)$$

• If $\alpha_1 - \alpha_2$ is integer then the second independent solution can be found in the form

$$R_2(r) = \underbrace{C\, R_1(r)}_{\text{constant} \quad \text{as in (8)}} \ln r + r^{\alpha_2}\left(\tilde{c}_0 + \tilde{c}_1 r^2 + \tilde{c}_2 r^3 + \dots\right)$$

Let us demonstrate how this theory works in the case of the Bessel equation

In the case of the Bessel equation the coefficients P, Q, S from (5) are

$$P(r) = r^2, \quad Q(r) = r, \quad S(r) = r^2 - n^2 = -n^2 + \underbrace{r^2}_{\text{the higher order term}}$$

$$\Downarrow$$

The corresponding Euler equation is

$$r^2 R'' + r R' - \nu^2 R = 0 \qquad (\ )$$

i.e. for ν integer it is exactly the same equation as we obtained when using the method of separation of variables for the Laplace equation on the disk (see Lecture 7, page 3, eq 5) so the indicial equation is the same:

$$\alpha(\alpha - 1) + \alpha - \nu^2 = 0 \iff \alpha^2 - \nu^2 = 0 \iff \alpha = \pm\nu$$

Without loss of generality assume that $\nu \geq 0$

According to the general scheme given on page 5 let us look for a solution in the form

$$R(r) = r^\nu (c_0 + c_1 r + c_2 r^2 + \dots) = \sum_{i=0}^{\infty} c_i r^{\nu + i}$$

Since we can differentiate power series term by term we have

$$R(r) = \sum_{i=0}^{\infty} c_i r^{\nu+i} \Rightarrow$$

$$R'(r) = \sum_{i=0}^{\infty} (\nu+i) c_i r^{\nu+i-1}$$

$$R''(r) = \sum_{i=0}^{\infty} (\nu+i)(\nu+i-1) c_i r^{\nu+i-2}$$

Substitute this into the Bessel eq. (2):

$$r^2 R'' + rR' + (r^2-\nu^2) R =$$

$$r^2 \sum_{i=0}^{\infty} (\nu+i)(\nu+i-1) c_i r^{\nu+i-2} + r \sum_{i=0}^{\infty} (\nu+i) c_i r^{\nu+i-1} +$$

$$+ (r^2-\nu^2) \sum_{i=0}^{\infty} c_i r^{\nu+i} = \sum_{i=0}^{\infty} \underbrace{\Big((\nu+i)(\nu+i-1) + (\nu+i) - \nu^2\Big)}_{(\nu+i)^2-\nu^2 = i(2\nu+i)} c_i r^{\nu+i} +$$

$$+ \underbrace{\sum_{i=0}^{\infty} c_i r^{\nu+i+2}}_{\sum_{i=2}^{\infty} c_{i-2} r^{\nu+i}} = \sum_{i=0}^{\infty} i(2\nu+i) c_i r^{\nu+i} + \sum_{i=2}^{\infty} c_{i-2} r^{\nu+i} = 0$$

Now compare coefficients:

1) if $i=0$ $\quad 0 \cdot c_0 = 0 \Rightarrow$ no conditions

2) if $i=1$ $\quad 1(2\nu+1) c_1 = 0 \Rightarrow \boxed{c_1 = 0}$ (8)

3) if $i \geq 2$ $\quad \underbrace{i(2\nu+i)}_{\text{always} > 0} c_i + c_{i-2} = 0 \Rightarrow$ we get

the following recurrence formula (i.e. how the ith term of the sequence c_i is expressed via the previous terms)

$$\boxed{c_i = -\frac{c_{i-2}}{i(2\nu+i)}} \quad (9)$$

So

$$c_2 = -\frac{c_0}{2\cdot(2\nu+2)} \quad (10)$$

$$c_3 = -\frac{c_1}{3\cdots} = 0 \quad \text{since } c_1 = 0$$

and in general $\boxed{c_i = 0 \text{ for odd } i}$ (11)

$$c_4 = -\frac{c_2}{4(2\nu+4)} \overset{(10)}{=} \frac{c_0}{2\cdot 4(2\nu+2)(2\nu+4)}$$

and in general

$$c_{2k} = \frac{(-1)^k c_0}{\underbrace{2\cdot 4\cdots 2k}_{2^k k!}\,\underbrace{(2\nu+2)(2\nu+4)\cdots(2\nu+2k)}_{2^k(\nu+1)\cdots(\nu+k)}} = \frac{(-1)^k c_0}{2^{2k}k!(\nu+1)\cdots(\nu+k)}$$

$$\Downarrow$$

$$R(r) = c_0 r^\nu \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k}k!(\nu+1)\cdots(\nu+k)}$$

The conventional choice of the constant c_0

If ν is nonnegative integer $c_0 := \frac{1}{2^\nu \nu!}$ =>

$$R(r) = \frac{r^\nu}{2^\nu \nu!} \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k}k!(\nu+1)\cdots(\nu+k)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+k)!}\left(\frac{r}{2}\right)^{2k+\nu}$$

There is a natural generalization of the factorial to noninteger numbers called Gamma function $\Gamma(\nu)$

$$\Gamma(\nu) := \int_0^\infty t^{\nu-1} e^{-t}\, dt \quad , \quad \mathrm{Re}\,\nu > 0$$

The main property of the Gamma function is that

- $\Gamma(\nu) = (\nu-1)\Gamma(\nu-1) \qquad (12)$

Indeed $\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t}\, dt \underset{\substack{\text{integration}\\\text{by parts}}}{=} \underbrace{-t^{\nu-1}e^{-t}\Big|_0^\infty}_{0} + (\nu-1)\int_0^\infty t^{\nu-2} e^{-t}\, dt =$

$= (\nu-1)\Gamma(\nu-1)$

- $\Gamma(1) = 1 \qquad \left(\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1 \right) \Rightarrow$

$\Gamma(2) = 1\,\Gamma(1) = 1$, $\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1$, $\Gamma(4) = 3\,\Gamma(3) = 3\cdot 2\cdot 1 =$

$= 3!$ and in general $\Gamma(\nu) = (\nu-1)!$ for natural ν.

So for arbitrary ν the conventional choice of the constant c_0 is $c_0 := \frac{1}{2^\nu \Gamma(\nu+1)} \Rightarrow$ with this c_0

Def The function

$$J_\nu(r) := \sum \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{r}{2}\right)^{\nu+2k}$$

is called the Bessel function of the first kind of order ν (converge for any r)

Rem If ν is not an integer then we can find a second independent solution in the form

$$r^{-\nu}\left(c_0 + c_1 r + c_2 \dots\right)$$

because in this case (replacing ν by $-\nu$) we have the recursive formula

$$i(-2\nu+i)c_i + c_{i-2} = 0 \quad (13), \quad c_1 = 0$$

If ν is not integer and ν is not a half integer then $i(-2\nu+i)$ is never zero then c_i can be found from c_{i-2} for any $i \geq 2$.

If ν is half of the odd integer for $i = 2\nu$ $i(-2\nu+i) = 0$ but we still fine because we can assume that $c_i = 0$ for all odd i.

However if ν is an integer then for $i = 2\nu$

the eq. (13) becomes inconsistent

$$0 \cdot c_i + c_{i-2} = 0 \quad \text{but } c_{i-2} \neq 0$$

So in this case we should find a solution

in a different form:

$$C\, J_\nu(r) \ln r + r^{-\nu}\left(c_0 + c_1 r + c_2 r^2 + \dots\right)$$

Note that in all case the second independent solution goes to infinity as $r \to 0$

Therefore $J_\nu(r)$ is the unique (up to the multiplication by a constant) solution of the Bessel equation (2) such that $J_\nu(0)$ is finite and we will use it below

Rem. There is a conventional choice of the second indep solution called the Bessel function of second kind of order ν, but we do not discuss it here

The Bessel function of the second kind are used when one considers boundary value problem on cylindrical domains with cylindrical holes.

Now let us return to our PDE's (see the table on page 1)

If $u(1,\theta,t)=0 \rightarrow$ Heat eq $\rightarrow$ Wave eq

$u(1,\theta,z)=0$ Laplace equation

Then the ODE for R is

$$r^2R''+rR'+(\lambda^2r^2-n^2)R=0$$

$R(0)$ is finite and $R(1)=0$

Finiteness of $R(0)$ and Proposition 1 of page 2 implies that $R(r)=c\,\underbrace{J_n(\lambda r)}_{\text{Bessel of order } n}$

$R(1)=0 \Leftrightarrow J_n(\lambda)=0$ i.e.

λ is ~~a~~ positive root of $J_n(r)=0$ or positive zero of $J_n(r)$

Rem The reason we from the beginning took $c>0$ on page 2 is that if $c<0$ $c=(i\lambda)^2$ then $J_n(i\lambda r)=0$ does not have positive roots)

It turns out (we do not prove it because of lack of time) that $J_n(r)=0$ has infinite many positive roots

$$\lambda_{n1}<\lambda_{n2}<\dots<\lambda_{nk}<$$

Also they satisfy the following properties

$$\lambda_{ni}<\lambda_{n+1\,i}<\lambda_{n\,i+1}$$

(i.e. the ith positive root of J_{n+1} is between the ith and $(i+1)$th root of J_n)

Also $\pi(i-1)<\lambda_{ni}<\pi(n+i) \Rightarrow \dfrac{\lambda_{ni}}{\pi i}\xrightarrow[i\to\infty]{}1$

Moreover $\lambda_{n\,i+1}-\lambda_{n,i}\xrightarrow[i\to\infty]{}\pi$, i.e. the difference between two consequtive zeros tends to π as the index goes to infinity. Moreover

$$J_\nu(r)\sim\gamma\frac{\sin(r+\delta)}{\sqrt{r}}+O\left(\frac{1}{r^{3/2}}\right)$$

It turns out that the sequence of functions

$J_n(\lambda_{n1} r), J_n(\lambda_{n2} r), \ldots, J_n(\lambda_{ni} r), \ldots$

where $\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \ldots, \lambda_{nn}, \ldots$ are positive zeros of the Bessel function $J_n(r)$ are orthogonal with respect to the following weighted inner product in the space $C(0,1)$

$$\langle f, g\rangle = \int_0^1 f(r)\, g(r) \underbrace{r\,dr}_{\text{weighted}}$$

(see appendix will be posted later)

Moreover their norm with respect to this inner product is (again see appendix)

$$\|J_n(\lambda_{ni} r)\|^2 = \int_0^1 \left(J_n(\lambda_{ni} r)\right)^2 r\,dr = \frac{1}{2}\left(J_{n+1}(\lambda_{ni})\right)^2$$

To any function f (say piecewise continuous) on $(0,1)$ one can assign the Fourier-Bessel series

$$f(r) = A_1 J_n(\lambda_{n1} r) + A_2 J_n(\lambda_{n2} r) + \ldots + A_k J_n(\lambda_{nk} r)$$

like Fourier series just instead of $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$ we use $J_n(\lambda_{n1} r), J_n(\lambda_{n2} r), \ldots J_n(\lambda_{nk} r)$

and since $\{J_n(\lambda_{ni} r)\}_{i=1}^{\infty}$ is an orthogonal set

$$A_i = \frac{1}{\|J_n(\lambda_{ni} r)\|^2} \langle f, J_n(\lambda_{ni} r)\rangle = \frac{2}{J_{n+1}(\lambda_{ni})^2} \int_0^1 f(r)\, J_n(\lambda_{ni} r)\, r\,dr$$

a) The most simple application: Heat equation of the circular plate with rim kept at zero temperature (and also faces insulated so that the heat flow on the plane of the plate only) and the initial temperature $f(r)$ depending on r only (not on θ) ∴ So we expect that our solution in cylindrical coordinates (r, θ, t) will be independent of θ as well

$$u_t = k(u_{xx} + u_{yy}) \Leftrightarrow u_t = k\left(u_{rr} + \frac{1}{r} u_r + \underbrace{\frac{1}{r^2} u_{\theta\theta}}_{0}\right)$$

$$u(1, t) = 0$$

$$u(r, 0) = f(r)$$

since u is independent of θ

$$u = T(t) R(r)$$

$$\frac{T'}{kT} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0 \rightarrow \text{Bessel equation of order } 0$$

$$T' = -k\lambda^2 T \Rightarrow T = e^{-k\lambda_{0i}^2 t}$$

all possible values of λ are λ_{0i} → zeros of the Bessel Function

Solutions of the form $R(r)T(t)$ are

$$u(r,t) = A_i J_0(\lambda_{0i} r) e^{-k\lambda_{0i}^2 t}$$

The solution satisfying $u(r, 0) = F(r)$ is found in the form of infinite sum

$$u(r,t) = \sum A_i e^{-k\lambda_{0i}^2 t} J_0(\lambda_{0i} r) \Rightarrow f(r) = \sum A_i J_0(\lambda_{0i} r)$$

where A_i are the Fourier-Bessel coefficient

of f (w.r.t. to the scaling of the Bessel function J_0)

$$A_i = \frac{2}{J_1(\lambda_{0i})^2} \int_0^1 f(r) J_0(\lambda_{0i} r) r\,dr$$

More complicated application for heat transfer of the circular plate

If $u(r,\theta,0) = f(r,\theta)$ - depends on θ.

$$\frac{T'}{kT} = \underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{separating further}} = -\lambda^2$$

$T' = -k\lambda^2 T \quad T = e^{-k\lambda_{ni}^2 t}$

$\frac{\Theta''}{\Theta} = -\alpha \rightarrow$ Θ is periodic implies that $\alpha = n^2 \rightarrow A\cos n\theta + B\sin n\theta$

$r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0 \rightarrow$ Bessel

$J_n(\lambda_{ni} r)$

The solution of the form $T(t)R(r)\,\Theta(\theta)$ is

$$e^{-k\lambda_{ni}^2 t} J_n(\lambda_{ni} r)(A\cos n\theta + B\sin\theta)$$

Then we look for the solution as infinite series

$$\boxed{\sum_{n=0}^{\infty}\sum_{i=1}^{\infty} e^{-k\lambda_{ni}^2 t} J_n(\lambda_{ni} r)(A_{ni}\cos n\theta + B_{ni}\sin n\theta)}$$

summed

If $t=0$ we want that

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} J_n(\lambda_{ni} r)(A_{ni}\cos n\theta + B_{ni}\sin n\theta)$$

$$\sum C_n(r)\cos n\theta + D_n(r)\sin n\theta \quad \text{where}$$

$$C_n(r) = \begin{cases} \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos n\theta\, d\theta, & n \geq 1 \\ \frac{1}{2\pi}\int^{2\pi} f(r,\theta)\cos n\theta\, d\theta, & n=0 \end{cases}$$

$$D_n(r) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin n\theta\, d\theta$$

$$C_n(r) = \sum_{k=1}^{\infty} A_{ni} J_n(\lambda_{ni} r)$$

$$D_n(r) = \sum_{k=1}^{\infty} B_{ni} J_n(\lambda_{ni} r)$$

$$A_{ni} = \frac{2}{J_{n+1}^2(\lambda_{ni})}\int_0^1 C_n(r) J_n(\lambda_{ni} r)\, r\, dr =$$

$$= \frac{2}{\pi J_{n+1}^2(\lambda_{ni})}\int_0^1\int_0^{2\pi} r f(r,\theta) J_n(\lambda_{ni} r)\cos n\theta\, d\theta\, dr \quad \text{for } n \geq 1$$

$$A_{0i} = \frac{1}{\pi J_1^2(\lambda_{0i})} \int_0^1 \int_0^{2\pi} r f(r,\theta) J_0(\lambda_{0i} r) \, d\theta \, dr$$

$$B_{ni} = \frac{2}{\pi J_{n+1}(\lambda_{ni})^2} \int_0^1 \int_0^{2\pi} r f(r,\theta) J_n(\lambda_{ni} r) \sin n\theta \, d\theta \, dr$$

c) Vibrating drum example

Assume that we have a drum with a circular head (of unit radius) and it is pushed a little bit such that its initial shape is $f(r,\theta)$ and then one lets it go (starting with zero as initial velocity). What is its shape in any time moment.

We have the following initial/boundary value problem: (in cylindrical coordinates)

$$u_{tt} = a^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$$

$u(1,\theta,t) = 0$ (the head is glued to the drum)

$u(r,\theta,0) = f(r,\theta)$ → initial shape

$u_t(r,\theta,0) = 0$

Using the first column of the table on page 1

$u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t) \Rightarrow$

$(c = \lambda^2)$

- $\Theta'' + n^2\Theta = 0$ (we use again the periodicity of $\Theta(\theta)$ with period 2π)

$\Rightarrow$ $\boxed{\Theta = A\cos n\theta + B\sin n\theta}$

- $r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0 \rightarrow$ again the Bessel eq. of order n

$R(0)$ is finite and, $R(1) = 0$ (because $u(1,\theta,t)=0$) $\Rightarrow$

$\lambda = \lambda_{ni}$ - the ith positive zero of the Bessel function $J_n(r)$ $\Rightarrow$

$\boxed{R(r) = cJ_n(\lambda_{ni} r)}$

- $T'' + a^2\lambda^2 T = 0$

$T'(0) = 0$ (because $u_t(r,\theta,0) = 0$) $\Rightarrow$

$\boxed{T(t) = C\cos(a\lambda_{ni} t)}$

So the separated solution $R(r)\Theta(\theta)T(t)$ has the form $J_n(\lambda_{ni} r)\cos(a\lambda_{ni} t)(A_{ni}\cos n\theta + B_{ni}\sin n\theta)$

$n \geq 0$, $i \geq 1$. and the solution that satisfies also $u(r,\theta,0) = f(r,\theta)$ is the infinite sum

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} J_n(\lambda_{ni} r)\cos(a\lambda_{ni} t)(A_{ni}\cos n\theta + B_{ni}\sin n\theta)$$

where A_{ni} and B_{ni} are exactly as on pages 17-18