Differential invariants of classical generic Monge–Ampère equations of hyperbolic type

Valeriy A. Yumaguzhin
Program Systems Institute of RAS, Pereslavl'-Zalesskiy, Russia
yuma@diffiety.botik.ru

A classical Monge–Ampère equation is a PDE of the form

$$N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0\,, \tag{1}$$

where the coefficients N, A, B, C, D are smooth functions of x, y, z, z_x, z_y.

It is well known that a most general invertible transformation of variables for equations (1) is a contact transformation and contact transformations preserve the class of all classical hyperbolic Monge–Ampére equations.

We represent some approach to calculate differential invariants of classical generic Monge–Ampére equations of hyperbolic type w.r.t. contact transformations and solve the equivalence problem for them. These results are obtained by A.M.Vinogradov, M.Marvan, and author in [1].

1. Monge–Ampère equations from geometric point of view.

By M we denote the space of the variables x, y, z, z_x, z_y considering as independent, by U_1 the standard contact form $dz - z_x dx - z_y dy$ on M, and by $\mathcal{C}$ the standard contact distribution on M that is the distribution $p \mapsto \mathcal{C}_p$, where $\mathcal{C}_p$ is the kernel of the form U_1 at $p \in M$. Recall that a diffeomorphism $f : M \to M$ preserving $\mathcal{C}$ is called a contact transformation.

Theorem 0.1. *Every classical hyperbolic Monge–Ampère equation $\mathcal{E}$ naturally determines a pair of 2-dimensional subdistributions*

$$\mathcal{D}^1 : p \mapsto \mathcal{D}^1_p\,, \quad \mathcal{D}^2 : p \mapsto \mathcal{D}^2_p$$

of the contact distribution $\mathcal{C}$ so that:

(1) $\mathcal{C}_p = \mathcal{D}^1_p \oplus \mathcal{D}^2_p$,

(2) *$\mathcal{D}^1_p$ and $\mathcal{D}^2_p$ are skew-orthogonal w.r.t. the symplectic form* $dU_1\big|_p$.

(3) *The pair $(\mathcal{D}^1, \mathcal{D}^2)$ reconstructs the equation $\mathcal{E}$ uniquely,*

(4) *The correspondence $\mathcal{E} \to (\mathcal{D}^1, \mathcal{D}^2)$ is a bijection between all classical hyperbolic Monge–Ampère equations and pairs of 2-dimensional skew-orthogonal subdistributions of $\mathcal{C}$.*

Thus, every hyperbolic Monge–Ampère equation $\mathcal{E}$ is naturally identified with the pair of 2-dimensional, skew-orthogonal subdistributions $(\mathcal{D}^1, \mathcal{D}^2)$ of the contact distribution $\mathcal{C}$ on M. In particular, the equivalence problem for these equations with respect to contact transformations is interpreted as the equivalence problem for corresponding pairs of 2-dimensional, skew-orthogonal subdistributions with respect to contact transformations.

2. Projections. Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation. By $(\mathcal{D}^i)^1$, $i = 1, 2$, we denote the distribution generated by all vector fields $X, Y \in \mathcal{D}^i$ and their commutators $[X, Y]$. Then we have

$$\dim(\mathcal{D}^1)^{(1)} = \dim(\mathcal{D}^2)^{(1)} = 3$$

and the distribution

$$D^3 = (\mathcal{D}^1)^{(1)} \cap (\mathcal{D}^2)^{(1)}$$

is 1-dimensional and transversal to $\mathcal{C}$. Therefore we get the decomposition of the tangent space $T(M)$ to M

$$T(M) = \mathcal{D}^1 \oplus \mathcal{D}^2 \oplus \mathcal{D}^3. \tag{2}$$

This decomposition generates the projections

$$\mathcal{P}_i : T(M) \to \mathcal{D}^i,\ i = 1,2,3, \quad \mathcal{P}_j^{(1)} : T(M) \to \mathcal{D}^j \oplus \mathcal{D}^3,\ j = 1,2\,.$$

These projections are interpreted as vector-valued 1-forms and they are differential invariants of $\mathcal{E}$ w.r.t. contact transformations.

3. Curvature forms of the distributions. By $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_1^1$ and $\mathcal{R}_2^1$ we denote the curvature forms of the distributions $\mathcal{D}^1$, $\mathcal{D}^2$, $(\mathcal{D}^1)^{(1)}$, and $(\mathcal{D}^2)^{(1)}$ respectively. Decomposition (2) allows to consider these curvature forms as a vector-valued differential 2-forms on M. They are differential invariants of $\mathcal{E}$ w.r.t. contact transformations.

4. Further differential invariants. Further invariants can by obtained just by applying various natural operations of tensor algebra, Frölicher–Nijenhuis brackets, etc. to the already obtained differential invariants. In particular, for a generic equation $\mathcal{E}$, five functionally independent scalar differential invariants $I^1, \ldots, I^5$ are obtained. Also, an invariant complete parallelism on M, that is a collection of five invariant vector fields $\{\, Y_1, \ldots, Y_5 \,\}$ linearly independent at every point of M, is obtained in this way.

5. The equivalence problem. The above-mentioned invariant vector fields Y_1, Y_2 and Y_3, Y_4 generate the distributions $\mathcal{D}^1$ and $\mathcal{D}^2$ respectively, that is $\mathcal{D}^1 = \langle Y_1, Y_2 \rangle$ and $\mathcal{D}^2 = \langle Y_3, Y_4 \rangle$. The above-mentioned scalar invariants $I^1, \ldots, I^5$ form a coordinate system in M. We say that the expression of $\mathcal{E} = \big(\langle Y_1, Y_2 \rangle,\ \langle Y_2, Y_4 \rangle\big)$ in this coordinate system is a *canonical form* of the equation $\mathcal{E}$.

Theorem 0.2. *Suppose $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are classical generic Monge–Ampère equations of hyperbolic type. Then $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are (locally) equivalent iff their canonical forms are the same.*

References

[1] A.M. Vinogradov, M. Marvan, and V.A. Yumaguzhin, *Differential invariants of generic hyperbolic Monge–Ampère equations* Russian Acad. Sci. Dokl. Math. **405** (2005) 299–301 (in Russian). English translation in: Doklady Mathematics **72** (2005) 883–885.