SISSA–INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

INTERNATIONAL WORKSHOP on

GEOMETRY OF VECTOR DISTRIBUTIONS, DIFFERENTIAL EQUATIONS, AND VARIATIONAL PROBLEMS

13–15 December 2006

(Room D, SISSA Main Building)

WORKSHOP ABSTRACTS

ABSTRACTS OF TALKS

HOMOGENEOUS BI-LAGRANGIAN MANIFOLDS OF SEMISIMPLE GROUP AND GENERALIZED GAUSS DECOMPOSITIONS

Dmitri Alekseevsky

University of Hull, UK

The talk is bases on a joint work with C.Medori (Parma).

A bi-Lagrangian structure on a (real or complex) symplectic manifold (M, ω) is a decomposition $TM = T^+ + T^-$ of the tangent bundle TM into a direct sum of integrable Lagrangian ($\omega|_{T^\pm} = 0$) subbundles $T^\pm$. In the real case, a manifold M with a bi-Lagrangian structure $(\omega, T^\pm)$ can be identified with a para-Kähler manifold (M, g, J) where $J \in \Gamma(\mathrm{End}TM)$ is the involutive endomorphism with ± 1-eigenspace distributions $T^\pm$ and $g = \omega \circ J$ is the pseudo-Riemannian metric such that the endomorphism J is g-skew-symmetric and parallel with respect to Levi-Civita connection of g.

The problem of classification of bi-Lagrangian manifolds $(M, \omega, T^\pm)$ which admit a semisimple transitive Lie group G of automorphisms reduces to a description of generalized Gauss decompositions

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}^+ + \mathfrak{m}^-$$

of the Lie algebra $\mathfrak{g}$ of G where $\mathfrak{p}^\pm := \mathfrak{k} + \mathfrak{m}^\pm$ are opposite parabolic subalgebras with the reductive part $\mathfrak{k}$ which is the stability subalgebra of $\mathfrak{g}$. We give a description of such decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ in terms of crossed Dynkin diagrams and a real semisimple Lie algebra $\mathfrak{g}$ in terms of crossed Satake diagrams.

PARABOLIC GEOMETRIES DETERMINED BY SUBBUNDLES IN THE TANGENT BUNDLE

Andreas Cap

University of Vienna and Erwin Schrodinger Institute of Mathematical Physics, Austria

The general theory of parabolic geometries can be used to obtain canonical Cartan connections associated to certain types of subbundles in the tangent bundle. Among the examples covered by this are generic distributions of rank 2 in dimension 5, rank 3 in dimension 6, and rank 4 in dimension 7. Apart from existence of Cartan connections, this theory also provides a number of efficient tools to study the geometry of such distributions. In my lecture, I will discuss some applications of these tools to questions of infinitesimal automorphisms and infinitesimal deformations of such structures.

EQUIVALENCE AND INVARIANTS OF SCALAR VARIATIONAL PROBLEM OF HIGHER ORDER

Boris Doubrov

Belorussian State University, Minsk, Belarussia

This is the joint work with Igor Zelenko.

We consider the geometry and equivalence problem of scalar Lagrangians $f(x, y, y', \ldots, y^{(n)})\, dx$ of order $n \geq 3$ viewed modulo contact transformations, divergence and multiplication to a non-zero constant. We show that this equivalence problem coincides with the equivalence problem of some subclass of rank two vector distributions.

Using the recently developed geometry of rank 2 vector distributions, we construct of the canonical coframe naturally associated with each such variational problem. In particular, we show that all maximally symmetric non-degenerate Lagrangians are equivalent to $\left(y^{(n)}\right)^2\, dx$. We also discuss the correspondence between symmetries and invariants of the variational problem, the associated rank 2 vector distribution, and the corresponding Euler–Lagrange equation.

DOUBROV INVARIANTS, TWISTOR THEORY AND EXOTIC HOLONOMY

Maciej Dunajski

University of Cambridge, UK

I shall review the twistor approach to ODEs for which the space of solutions admits a splitting of the tangent bundle as a symmetric tensor product of rank-two vector bundles. This condition leads to the vanishing of relative invariants originally due to Doubrov. The special case of 4th order ODEs leads to exotic holonomy of Bryant. Despite this technical description it will be an elementary talk aimed at audience with no knowledge of twistor theory, but some familiarity with complex analysis.

REALIZABLE GROWTH VECTORS AND NORMALIZATION OF HOMOGENEOUS APPROXIMATIONS OF AFFINE CONTROL SYSTEMS

Svetlana Ignatovich

Kharkov National University, Ukraine

In my talk, I will discuss some problems arising in connection with the study of homogeneous approximations of control systems.

More precisely, we consider affine control systems $\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$, where $X_i(x)$ are real analytic in a neighborhood of the origin. The object under consideration is a *growth vector* (at the origin of such system. *The first ("realizability") problem* is: for a given sequence $v = (v_1, \ldots, v_p)$, to determine if there exists a system having the growth vector v.

Obviously, if two systems have the same homogeneous approximation then they have the same growth vector. Thus, *the second problem* is: for a given growth vector v to describe all possible homogeneous approximations of systems with this growth vector.

Observe that (nonsingular) changes of the control can change the homogeneous approximation (however, they cannot change the growth vector). Hence, *the third problem* is: for a given growth vector, to normalize (if it is possible) homogeneous approximations using changes of the control, and to describe all possible normal forms.

The lecture is based on a joint work with A.Agrachev.

INVARIANTS OF VECTOR DISTRIBUTIONS OF CORANK 2

Bronislaw Jakubczyk

Stefan Banach International Mathematical Center,Warsaw, Poland

We shall discuss equivariants and invariants of vector distributions $D \subset TM$ of corank 2 on even-dimensional manifolds. We shall assign, to a given distribution D, a field of characteristic lines in the annihilator $D^{\perp}$, as well as a field of characteristic lines in D. These fields have canonical normalizations, which yield fields of normalized characteristic 1-forms and normalized characteristic vector fields. The equivalence problem for such distributions is reduced to equivalence of tuples of differential 1-forms or tuples of vector fields. We give explicit criteria for equivalence of generic distributions. In particular, we construct explicitely functional invariants.

GENERIC RESULTS FOR HORIZONTAL AND SINGULAR CURVES OF RANK-VARYING DISTRIBUTIONS

Frederic Jean

ENSTA, Paris, France

In this talk, we provide characterizations for singular curves of rank-varying distributions. We prove that, under generic assumptions, such curves share nice properties, related to computational aspects; more precisely, we show that, for a generic rank-varying distributions (with respect to the Whitney topology), all nontrivial singular curves are of minimal order and of corank one. As a consequence, for a Riemannian manifold (M, g) and for a generic rank-varying distribution D of dimension at least 3, the sub-Riemannian manifold (M, D, g) does not admit nontrivial minimizing singular curves. We also prove that, given a rank-varying distribution, singular curves are strictly abnormal, generically with respect to the Riemannian metric. We then show how these results can be used to derive regularity results for the distance function and in the theory of Hamilton-Jacobi equations.

INVARIANTS OF ALMOST-PRODUCT STRUCTURES AND GEOMETRY OF MONGE-AMPERE AND JACOBI EQUATIONS

Aleksei Kushner

Astrakhan State University and Control Sciences Institute of the Russian Academy of Sciences, Moscow, Russia

Let N be a smooth manifold and let $\mathcal{P} = (P_1, \ldots, P_r)$ be an ordered set of real (or complex) distributions on N, i.e.

$$P_i : N \ni a \mapsto P_i(a) \subset T_aN \quad (\text{or } T_aN^{\mathbb{C}}).$$

The set $\mathcal{P}$ is called an *almost product structure* or a *$\mathcal{P}$-structure* [1] on N if at each point $a \in N$ the tangent space T_aN (for real distributions) or its complexification $T_aN^{\mathbb{C}}$ (for complex ones) splits in the direct sum of the subspaces $P_1(a), \ldots, P_r(a)$, i.e. $T_aN = \oplus_{i=1}^{r} P_i(a)$ or $T_aN^{\mathbb{C}} = \oplus_{i=1}^{r} P_i(a)$.

We get the following decomposition of the de Rham complex: the $C^{\infty}(N)$-modules of differential s-forms $\Omega^s(N)$ split in the direct sum

$$\Omega^s(N) = \bigoplus_{|\mathbf{k}|=s} \Omega^{\mathbf{k}}(N), \tag{0.1}$$

where $\mathbf{k}$ is a multiindex, $\mathbf{k} = (k_1, \ldots, k_r)$, $k_i \in \{0, 1, \ldots, n_i\}$, $n_i = dimP_i$, $|\mathbf{k}| = k_1 + \cdots + k_r$,

$$\Omega^{\mathbf{k}}(N) = \otimes_{i=1}^{r} \Omega$$

and

$$\Omega^{k_i}(P_i) = \left\{ \alpha \in \Omega^{k_i}(N) \middle| X \rfloor \alpha = 0 \; \forall X \in D(P_1) \oplus \cdots \oplus \widehat{D(P_i)} \oplus \cdots \oplus D(P_r) \right\}.$$

In case of complex almost product structures we have to consider the complexification $\Omega^s(N)^{\mathbb{C}}$ of the module $\Omega^s(N)$.

The de Rham differential d splits in the following direct sum:

$$d = \bigoplus_{|\mathbf{t}|=1} d_{\mathbf{t}},$$

where $t_j \in I_j = \{ z \in \mathbb{Z} | \, |z| \leq \dim P_j \}$ and

$$d_{\mathbf{t}} : \Omega^{\mathbf{k}}(N) \to \Omega^{\mathbf{k}+\sigma}(N).$$

Theorem 1. *If one of the component t_i of a multi-index $\mathbf{t}$ is negative, then operator $d_{\mathbf{t}}$ is a $C^\infty(N)$-homomorphism.*

In the other words, if one of the components t_i of the multiindex $\mathbf{t}$ is negative, then operator $d_{\mathbf{t}}$ is tensor which acts from $\Omega^{\mathbf{k}}(N)$ to $\Omega^{\mathbf{k}+\mathbf{t}}(N)$. Such tensors are invariant with respect to diffeomorphisms.

Any (hyperbolic or elliptic) classical Monge-Ampère equation on a two-dimensiomnal manifold M

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D\left(v_{xx}v_{yy} - v_{xy}^2\right) + E = 0. \tag{0.2}$$

is an almost product structure (real or complex) on the manifold of 1-jets $J^1(M)$ (see [1]).

Jacobi equations [1]

$$\begin{cases} A_1 + B_1\dfrac{\partial v_1}{\partial x} - C_1\dfrac{\partial v_1}{\partial y} - D_1\dfrac{\partial v_2}{\partial y} + E_1\dfrac{\partial v_2}{\partial x} + F_1 \det J_v = 0, \\ A_2 + B_2\dfrac{\partial v_1}{\partial x} - C_2\dfrac{\partial v_1}{\partial y} - D_2\dfrac{\partial v_2}{\partial y} + E_2\dfrac{\partial v_2}{\partial x} + F_2 \det J_v = 0, \end{cases}$$

can be regarded as almost product structures on a 4-dimensional manifold. Here

$$\det J_v = \begin{vmatrix} \dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_1}{\partial y} \\ \dfrac{\partial v_2}{\partial y} & \dfrac{\partial v_2}{\partial x} \end{vmatrix}$$

is a Jacobian and $A_i, B_i, C_i, D_i, E_i, F_i$ $(i = 1, 2)$ are some functions on x, y, v.

We applly constructed tensors to the problem of classification of Monge-Ampère and Jacobi equations solve the problem linearization of ones.

References

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RIGIDITY AND FLEXIBILITY OF HOMOGENEOUS VARIETIES

Joseph Landsberg

Texas A&M University , USA

I will describe a unified perspective for studying the Schubert rigidity of cycles in compact Hermitian symmetric spaces and the Griffiths-Harris rigidity of homogeneous subvarieties of projective space. This is joint work with C. Robles, building on work of Griffiths and Harris, Bryant and Hong.

REPRESENTATIONS OF GRADED LIE ALGEBRAS AND DIFFERENTIAL EQUATIONS ON FILTERED MANIFOLDS

Tohru Morimoto

Nara Women University, Japan

If we generalize the notion of a manifold to that of a filtered manifold, the usual rôle of tangent space is played by the nilpotent graded Lie algebra which is defined at each point of the filtered manifold as its first order approximation. On the basis of this nilpotent approximation we have been studying various structures and objects on filtered manifolds to develop Nilpotent Geometry and Analysis.

In this talk we present a simple principle to associate systems of differential equations to a representation of a Lie algebra in the framework of nilpotent analysis.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a transitive graded Lie algebra, that is, a Lie algebra satisfying:

i) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$
ii) $\dim \mathfrak{g}_- < \infty$, where $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$, the negative part of $\mathfrak{g}$
iii) (Transitivity) For $i \geq 0, x_i \in \mathfrak{g}_i$, if $[x_i, \mathfrak{g}_-] = 0$, then $x_i = 0$.

Let $V = \bigoplus_{q\in\mathbb{Z}} V_q$ be a graded vector space satisfying:

i) $\dim V_q < \infty$.
ii) There exists q_I such that $V_q = 0$ for $q \leq q_I$.

Let $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V such that

i) $\lambda(\mathfrak{g}_p)V_q \subset V_{p+q}$.
ii) There exists q_0 such that if $\lambda(\mathfrak{g}_-)x_q = 0$ for $q > q_0$ then $x_q = 0$.

We then consider the cohomology group $H(\mathfrak{g}_-, V) = \bigoplus_{p,r\in\mathbb{Z}} H^p_r(\mathfrak{g}_-, V)$ of the representation of $\mathfrak{g}_-$ on V, namely the cohomology group of the cochain complex:

$$\xrightarrow{\partial} \mathrm{Hom}(\wedge^{p-1}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p+1}\mathfrak{g}_-, V)_r \xrightarrow{\partial}$$

where $\mathrm{Hom}(\wedge^p \mathfrak{g}_-, V)_r$ is the set of all homogeneous p-cochain ω of degree r, that is, $\omega(\mathfrak{g}_{a_1} \wedge \cdots \wedge \mathfrak{g}_{a_p}) \subset V_{a_1+\cdots+a_p+r}$ for any $a_1, \cdots, a_p < 0$.

Now our assertion may be roughly stated as follows:

Principle 1. *The first cohomology group $H^1(\mathfrak{g}_-, V) = \bigoplus H^1_r(\mathfrak{g}_-, V)$ represents a system of differential equations and $V = \bigoplus V_q$ represents its solution space.*

We will explain that it is in the framework of nilpotent analysis that the principle above is properly and well settled. We will also give several examples.

Key words : filtered manifold, weighted jet bundle, geometric structure and differential equation on a filtered manifold.

HOW TO STRATIFY SPECIAL MULTI-FLAG AND GEOMETRICALLY ENCODE THE EMERGING STRATA

Piotr Mormul

Warsaw University, Poland

Classical Goursat distributions have been satisfactorily understood only when visualised on monster manifolds, as the outcomes of series of Cartan's prolongations (started from the tangent bundle to a 2-surface). Kumpera-Ruiz coordinates – that serve as unequalled night glasses for Goursat flags (and are much older than the monster constructions) – are then becoming more than natural. The KR coordinates on monsters allow the very basic *Kumpera-Ruiz classes* of Goursat germs defined in [1] to be re-examined and, in a sense, rediscovered in the process of prolongations.

Likewise, special multi-flags start to be properly understood when viewed on [even bigger than for Goursat] Monster Manifolds, resulting from series of *generalized* Cartan prolongations (gCp) described in [2]. On the other hand, it is known for some years now that special multi-flags are best visible in so-called Extended Kumpera-Ruiz coordinates. It so happens that the EKR coordinates are a matter of course on Monsters – just the first thing coming to mind, given the gCp operation. Whence the question about analogues for special multi-flags of KR classes/Goursat.
Such analogues can be defined right out of the EKR coordinates on Monsters (*reversing* the order of observations made, in the course of years, for Goursat objects). These are the EKR classes constructed in [2]. Our first objective is to present them in detail during the Workshop.

The EKR classes have been put on a solid geometric footing only later in [3], and termed *singularity classes*. Discussing that footing is cardinal, albeit time-consuming, cf. [4]. Instead –

and this is our second objective – we want to propose a super-encoding of the strata of obtained stratification(s). It is geometry itself, seconded by Lie algebra, that (super-)encodes every single singularity class. At present we do not know if that encoding is injective in all widths and lengths, for all singularity classes; only believe in its being injective. This is a standing open question.

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$GL(2,\mathbb{R})$ GEOMETRIES AND ODEs

Pawel Nurowski

Warsaw University, Poland

We will discuss the description of classes of contact equivalent ordinary differential equations of order $n \geq 5$ in terms of Cartan bundle B with the base being the space of solutions and with the structure group $\mathrm{GL}(2,\mathbb{R})$ semidirect product $\mathbb{R}^n$. In case of ODEs of order 5 we give necessary and sufficient conditions for contact equivalent classes of ODEs to define a Cartan connection on B with values in the Lie algebra of $\mathrm{GL}(2,\mathbb{R})$ semidirect product of $\mathbb{R}^5$. Relations between this geometry and recently developed SO(3) geometry in dimension 5 will also be discussed.

AVERAGE CONTROL SYSTEM AND FINSLER GEOMETRY

Jean-Baptiste Pomet

INRIA, Sophia Antipolis, France

This is the joint work with Alex Bombrun.

This talk will define an "average control system" for system that have a conservative drift and a very small control. The prototype is low thrust orbital transfer of earth satellites. Using averaging to treat small perturbations of integrable systems is not new; the originality here is that averaging can be performed before the variations of the control are decided, thus yielding really an "average control system". Under some rank assumptions, that are satisfied in the case of low thrust transfer, the new control system defines a norm in each tangent subspace. It is not in general twice differentiable, so that the term "finsler geometry" is not quite appropriate, but we shall present preliminary results.

THE CONFORMAL KILLING EQUATION ON FORMS - PROLONGATIONS AND APPLICATIONS

Joseph Silhan

Masarryk University in Brno, Czech Republic

We construct a conformally invariant vector bundle connection such that its equation of parallel transport is a first order system that gives a prolongation of the conformal Killing equation on differential forms. Parallel sections of this connection are related bijectively to solutions of the conformal Killing equation. We construct other conformally invariant connections, also giving prolongations of the conformal Killing equation, that bijectively relate solutions of the conformal Killing equation on k-forms to a twisting of the conformal Killing equation on $(k-\ell)$-forms for various integers ℓ. These tools are used to develop a helicity raising and lowering construction in the general setting and on conformally Einstein manifolds. (Joint work with A. Rod Gover.)

INCLUSIONS OF PARABOLIC GEOMETRIES ON A MANIFOLD

Jan Slovak

Masaryk University in Brno, Czech Republic

The lecture reports on joint research with Boris Doubrov aiming at the classification and study of all Fefferman type constructions for parabolic geometries where the underlying manifolds do not change.

The original Fefferman's construction produces an S^1–bundle over each manifold with an integrable CR–structure, equipped with a conformal structure. This construction corresponds to the embedding of Lie algebras $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ into $\tilde{\mathfrak{g}} = \mathfrak{so}(2p+2, 2q+2)$. Recently, two constructions appeared which may be interpreted as instances of such a procedure with the property that the underlying manifold does not change. Indeed conformal structures naturally associated with non-degenerate rank 2 vector distributions on 5-dimensional manifolds were studied by Nurowski and those associated with non-degenerate rank 3 distributions on 6-dimensional manifolds by Bryant. In both cases there are natural parabolic geometries associated with these distributions, which serve as an intermediate structure between the distribution and the conformal geometry.

Using classical results by A. Onischchik, we classify all possibilities of such inclusions of parabolic geometries. Apart of known examples, a new series of embeddings of 2–graded C_ℓ geometries into 1–graded $D_{\ell+1}$ geometries has been detected. These geometries correspond to the generic ℓ–dimensional distributions of codimension $\frac{1}{2}\ell(\ell-1)$ and the Bryant's example fits into this series with $\ell = 3$.

CLASSIFICATION PROBLEM AND SCALAR DIFFERENTIAL INVARIANTS

Alexandre Vinogradov

University of Salerno, Italy

A general scheme of solution of classification and equivalence problems for geometrical structures will be presented and illustrated by some examples (plane webs, Monge-Ampere equations, Ricci flat metrics).

CANONICAL FRAMES FOR VECTOR DISTRIBUTIONS OF RANK TWO AND THREE

Igor Zelenko

SISSA, Trieste, Italy

This is the joint work with Boris Doubrov.

First we will describe a new rather effective procedure of symplectification for the problem of local equivalence of non-holonomic vector distributions. The starting point of this procedure is to lift a distribution D to a special submanifold W_D of the cotangent bundle, foliated by the characteristic curves (the abnormal extremals of the distribution D). In particular, if D is a rank 2 distribution then the submanifolds W_D is nothing but the annihilator of the square of D, while if D is a distribution of odd rank it is the annihilator of D itself. The dynamics of the lifting (to W_D) of the distribution D along the characteristic curves (of W_D) is described by certain curves of flags of isotropic and coisotropic subspaces in a linear symplectic space. So, the problem of equivalence of distributions can be essentially reduced to the differential geometry of such curves: the invariants of these curves are automatically invariants of the distribution D and the canonical frame bundles, associated with such curves, can be in many cases effectively used for the construction of the canonical frames of the distributions D itself on certain fiber bundles over W_D. In this way we succeeded to construct the canonical frames for germs of rank 2 distributions in $\mathbb{R}^n$ with $n > 5$ and of rank 3 distributions in $\mathbb{R}^7$ from certain generic classes. The first case generalizes the classical work of E. Cartan (1910) on rank 2 distributions in $\mathbb{R}^5$. The second case is also new: the only rank 3 distributions, treated before, were rank 3 distributions in $\mathbb{R}^5$ (Cartan, 1910) and in $\mathbb{R}^6$ (N. Tanaka school and independently R. Bryant in 70th). In all these cases the most symmetric models will be given as well.

EXACT NORMAL FORM FOR (2,5) DISTRIBUTIONS

Michail Zhitomirskii

Technion -Israel Institute of Technology, Israel

I will present a complete solution of the classical problem on reduction of generic $(2,5)$ distributions to a normal form whose parameters are a complete system of independent invariants. The starting points is as follows: the Cartan invariant is a complete invariant in the classification of 3-quasi-jets of $(2,3,5)$ distributions with respect to the natural weights $1,1,2,3,3$.

RELATIVE LOCAL STABILITY OF FOR ENGEL STRUCTURES

Jiro Adachi
Hokkaido University, Sapporo, Japan

It is well known that contact structures are locally stable (the Darboux theorem).

The germ of contact structures at a point in manifolds of the same dimension are the same up to local diffeomorphisms. The local stability of contact structures relative to submanifolds was studied by Arnold and Givental.

This result was generalized by Zhitomirskii. He studied the local stability of contact structures relative to subsets which may not be manifolds. On the other hand, it is known that Engel structures are locally stable. In this talk we study relative local stability of Engel structures.

AN ESTIMATE FOR THE ENTROPY OF HAMILTONIAN FLOWS

Francesca Chittaro
SISSA, Trieste, Italy

We present a generalization to Hamiltonian flows on symplectic manifolds of the estimate proved by Ballmann and Wojtkovski for the dynamical entropy of the geodesic flow on a compact Riemannian manifold of nonpositive sectional curvature. Given such a Riemannian manifold M, Ballmann and Wojtkovski proved that the dynamical entropy h_μ of the geodesic flow on M satisfies the following inequality:

$$h_\mu \geq \int_{SM} \mathrm{Tr}\sqrt{-K(v)}\, d\mu(v),$$

where v is a unit vector in T_pM, if p is a point in M, SM is the unit tangent bundle on M, $K(v)$ is defined as $K(v) = \mathcal{R}(\cdot, v)v$, with $\mathcal{R}$ Riemannian curvature of M, and μ is the normalized Liouville measure on SM.

We consider a symplectic manifold M of dimension $2n$, and a compact submanifold N of M, given by the regular level set of a Hamiltonian function on M; moreover we consider a smooth Lagrangian distribution on N, and we assume that the reduced curvature $\hat{R}^h_z$ of the Hamiltonian vector field $\vec{h}$ with respect to the distribution is nonpositive. Then we prove that under these assumptions the dynamical entropy h_μ of the Hamiltonian flow w.r.t. the normalized Liouville measure on N satisfies:

$$h_\mu \geq \int_N \mathrm{Tr}\sqrt{-\hat{R}^h_z}\, d\mu.$$

EXPLICIT FORMULAS FOR BIHARMONIC SUBMANIFOLDS IN 3-DIMENSIONAL SASAKIAN SPACE FORMS

Dorel Fetcu

Technical University of Iassy, Romania

This is the joint work with Cezar Oniciuc.

Explicit formulas for biharmonic Legendre curves and biharmonic Hopf cylinders in 3-dimensional unit sphere endowed with a certain Sasakian structure are given. Parametric equations for biharmonic Hopf cylinders in Bianchi-Cartan-Vranceanu model spaces of a Sasakian space form are obtained.

QUATERNIONIC ANALOG OF CR GEOMETRY

Hiroyuki Kamada

Miyagi University of Education, Japan and University of Southern Denmark, Odense, Denmark

I will introduce quaternionic analogues of CR and pseudohermitian structures and their strong pseudoconvexity. Furthermore, I will explain the existence of a connection for quaternionic pseudohermitian structure, which stands for an analogue of the Tanaka-Webster connection, under certain conditions for convexity. On the other hand, O. Biquard gave the notion of quaternionic contact structure, which is another quaternionic analogue of CR structure. I will also explain difference between our structure and Biquard's one. This is a joint work with Shin Nayatani.

CHARACTERISTIC VECTOR FIELDS OF DISTRIBUTIONS - DETERMINATION THEOREMS

Woichech Krynski

Banach Mathematical Center, Poland

In our presentation we consider generic distributions $D \subset TM$ of corank $k \geq 2$ on manifolds M of dimension $n \geq 5$. We show that singular curves of such distribution determine the distribution on the subset of M where they generate at least two different directions. In particular, this happens on the whole of M if rank of D is odd. The distribution is determined by characteristic vector fields and their Lie brackets of appropriate order. We characterize pairs of vector fields of a corank 2 distribution. the case $k \geq 3$ is based on [2], the case $k = 2$ is based on [1].

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ON THE GEOMETRY OF HYPERSURFACES OF CONULLITY TWO IN EUCLIDEAN SPACES

Velichka Milousheva

Institute of Mathematics and Informatics, Bulgarian Academy of Science

We give a geometric description of the class $\mathfrak{K}_o$ of hypersurfaces of conullity two with involutive geometric two-dimensional distributions proving that the integral surfaces of these distributions are surfaces with flat normal connection, which are not developable and conversely, that any two-dimensional surface with flat normal connection, which is not developable, generates a hypersurface of conullity two from the class $\mathfrak{K}_o$. In this way the hypersurfaces of conullity two from the class $\mathfrak{K}_o$ are in one-to-one correspondence with the two-dimensional surfaces with flat normal connection, which are not developable.

We characterize the hypersurfaces of conullity two also as envelopes of two-parameter families of hyperplanes, proving that a hypersurface in Euclidean space is locally a hypersurface of conullity two if and only if it is the envelope of a two-parameter family of hyperplanes. This geometric characterization allows us to obtain a parametrization of each hypersurface of conullity two by a pair of a unit vector function $l(u, v)$ and a scalar function $r(u, v)$. We obtain a characterization in terms of a system of partial differential equations for the geometric functions $l(u, v)$ and $r(u, v)$ of two main classes of hypersurfaces of conullity two: minimal hypersurfaces of conullity two and hypersurfaces of conullity two of umbilical type.

GENERALIZED TRIANGULAR FORMS: COORDINATE-FREE DESCRIPTION AND BACKSTEPPING

Svyatoslav Pavlichkov

Kharkov National University

We investigate a new class of nonlinear control systems of O.D.E. which are not feedback linearizable in general. Our class is a generalization of the well-known feedback linearizable systems, and moreover it is a generalization of the triangular (or pure-feedback) forms studied before. We describe our "generalized triangular form" in coordinate-free terms of certain nested integrable distributions. Therefore, the problem of the feedback equivalence of a system to our generalized triangular form is solved in the whole state space by the definition of our class. We apply a specific backstepping procedure, and solve the problem of global controllability for our class. Our "backstepping algorithm", in turn, is based on the construction a certain discontinuous feedback law.

We propose to treat our class as a new canonical form which is a nonlinear global analog of the Brunovsky canonical form on the one hand, and is a global and coordinate-free generalization of the triangular form on the other hand.

DETERMINACY OF AFFINE DISTRIBUTIONS BY THEIR SINGULAR CURVES

Marek Rupniewski

Banach Mathematical Center, Poland

e consider control-affine systems of corank grater or equal 2 defined on a smooth manifold of dimension n, i.e., systems of the form:

$$\dot{q} = f_0 + u_1 f_1 + ... + u_k f_k, \quad k < n - 1.$$

We prove that if k is even then a generic system of this type is determined by its singular curves at every points belonging to some open and dense set. For odd k it may happen that there are no singular curves passing through an open set. Still, the above kind of determinacy holds for an open and dense subset of points through which at least one singular curve passes. In terms of distributions we prove that every generic germ of an affine distribution is determined by its singular curves. This is an analogue of the results obtained for linear distributions by Montgomery, Krynski, Jakubczyk-Krynski-Pelletier. Our results reduce some problems of classification of affine distributions, e.g. classification of rank 4 affine distributions on 6-dimensional manifold is reduced to classification of rank 2 affine distributions of special type.

AN EXPLICIT FORMULA FOR NUROWSKIS CONFORMAL STRUCTURE

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GAUSS-BONNET INEQUALITIES AND HOPF THEOREM FOR ZERMELOS PROBLEM

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DIFFERENTIAL INVARIANTS OF CLASSICAL GENERIC MONGE-AMP'ERE EQUATIONS OF HYPERBOLIC TYPE

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A classical Monge–Ampère equation is a PDE of the form

$$N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0\,, \tag{0.3}$$

where the coefficients N, A, B, C, D are smooth functions of x, y, z, z_x, z_y.

It is well known that a most general invertible transformation of variables for equations (0.3) is a contact transformation and contact transformations preserve the class of all classical hyperbolic Monge–Ampére equations.

We represent some approach to calculate differential invariants of classical generic Monge–Ampére equations of hyperbolic type w.r.t. contact transformations and solve the equivalence problem for them. These results are obtained by A.M.Vinogradov, M.Marvan, and author in [1].

1. Monge–Ampère equations from geometric point of view.

By M we denote the space of the variables x, y, z, z_x, z_y considering as independent, by U_1 the standard contact form $dz - z_x dx - z_y dy$ on M, and by $\mathcal{C}$ the standard contact distribution on M that is the distribution $p \mapsto \mathcal{C}_p$, where $\mathcal{C}_p$ is the kernel of the form U_1 at $p \in M$. Recall that a diffeomorphism $f : M \to M$ preserving $\mathcal{C}$ is called a contact transformation.

Theorem 0.1. *Every classical hyperbolic Monge–Ampère equation $\mathcal{E}$ naturally determines a pair of 2-dimensional subdistributions*

$$\mathcal{D}^1 : p \mapsto \mathcal{D}^1_p\,, \quad \mathcal{D}^2 : p \mapsto \mathcal{D}^2_p$$

of the contact distribution $\mathcal{C}$ so that:

i) $\mathcal{C}_p = \mathcal{D}^1_p \oplus \mathcal{D}^2_p$,

ii) $\mathcal{D}^1_p$ *and* $\mathcal{D}^2_p$ *are skew-orthogonal w.r.t. the symplectic form* $dU_1\big|_p$.

iii) *The pair* $(\mathcal{D}^1, \mathcal{D}^2)$ *reconstructs the equation* $\mathcal{E}$ *uniquely,*

iv) *The correspondence* $\mathcal{E} \to (\mathcal{D}^1, \mathcal{D}^2)$ *is a bijection between all classical hyperbolic Monge–Ampère equations and pairs of 2-dimensional skew-orthogonal subdistributions of* $\mathcal{C}$.

Thus, every hyperbolic Monge–Ampère equation $\mathcal{E}$ is naturally identified with the pair of 2-dimensional, skew-orthogonal subdistributions $(\mathcal{D}^1, \mathcal{D}^2)$ of the contact distribution $\mathcal{C}$ on M. In particular, the equivalence problem for these equations with respect to contact transformations is interpreted as the equivalence problem for corresponding pairs of 2-dimensional, skew-orthogonal subdistributions with respect to contact transformations.

2. Projections. Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation. By $(\mathcal{D}^i)^1$, $i = 1, 2$, we denote the distribution generated by all vector fields $X, Y \in \mathcal{D}^i$ and their commutators $[X, Y]$. Then we have

$$\dim(\mathcal{D}^1)^{(1)} = \dim(\mathcal{D}^2)^{(1)} = 3$$

and the distribution

$$D^3 = (\mathcal{D}^1)^{(1)} \cap (\mathcal{D}^2)^{(1)}$$

is 1-dimensional and transversal to $\mathcal{C}$. Therefore we get the decomposition of the tangent space $T(M)$ to M

$$T(M) = \mathcal{D}^1 \oplus \mathcal{D}^2 \oplus \mathcal{D}^3. \tag{0.4}$$

This decomposition generates the projections

$$\mathcal{P}_i : T(M) \to \mathcal{D}^i,\ i = 1, 2, 3, \quad \mathcal{P}^{(1)}_j : T(M) \to \mathcal{D}^j \oplus \mathcal{D}^3,\ j = 1, 2\,.$$

These projections are interpreted as vector-valued 1-forms and they are differential invariants of $\mathcal{E}$ w.r.t. contact transformations.

3. Curvature forms of the distributions. By $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}^1_1$ and $\mathcal{R}^1_2$ we denote the curvature forms of the distributions $\mathcal{D}^1$, $\mathcal{D}^2$, $(\mathcal{D}^1)^{(1)}$, and $(\mathcal{D}^2)^{(1)}$ respectively. Decomposition (0.4) allows to consider these curvature forms as a vector-valued differential 2-forms on M. They are differential invariants of $\mathcal{E}$ w.r.t. contact transformations.

4. Further differential invariants. Further invariants can by obtained just by applying various natural operations of tensor algebra, Frölicher–Nijenhuis brackets, etc. to the already obtained differential invariants. In particular, for a generic equation $\mathcal{E}$, five functionally independent scalar differential invariants $I^1, \ldots, I^5$ are obtained. Also, an invariant complete parallelism on M, that is a collection of five invariant vector fields $\{ Y_1, \ldots, Y_5 \}$ linearly independent at every point of M, is obtained in this way.

5. The equivalence problem. The above-mentioned invariant vector fields Y_1, Y_2 and Y_3, Y_4 generate the distributions $\mathcal{D}^1$ and $\mathcal{D}^2$ respectively, that is $\mathcal{D}^1 = \langle Y_1, Y_2 \rangle$ and $\mathcal{D}^2 = \langle Y_3, Y_4 \rangle$. The above-mentioned scalar invariants $I^1, \ldots, I^5$ form a coordinate system in M. We say that the expression of $\mathcal{E} = \big(\langle Y_1, Y_2 \rangle,\ \langle Y_2, Y_4 \rangle \big)$ in this coordinate system is a *canonical form* of the equation $\mathcal{E}$.

Theorem 0.2. *Suppose $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are classical generic Monge–Ampère equations of hyperbolic type. Then $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are (locally) equivalent iff their canonical forms are the same.*

References

[1] A.M. Vinogradov, M. Marvan, and V.A. Yumaguzhin, *Differential invariants of generic hyperbolic Monge–Ampère equations* Russian Acad. Sci. Dokl. Math. **405** (2005) 299–301 (in Russian). English translation in: Doklady Mathematics **72** (2005) 883–885.