SOLUTION OF PROBLEM 3 FROM ASSIGNMENT 1 IN ADVANCED CALCULUS I (MATH 409)

Problem: Prove that in the axioms of an ordered field, given in the class (and also in the textbook on p.4), the multiplicative property of order can be replaced by the following property:

$$a > 0,\ b > 0 \ \Rightarrow\ ab > 0 \tag{1}$$

(i.e. with this replacement we get an equivalent system of axioms).

Solution The fact that the multiplicative property of order implies (1) was proved in the class (Lecture 2, Consequences of the order axioms, item $2°$).

Let us prove that (1) together with other axioms of an ordered field (except the multiplicative property of order) imply the multiplicative property of order.

Case 1 Assume that $a < b$ and $c > 0$. By the distributive law $bc - ac = (b-a)c$. By our assumptions $b - a > 0$ and $c > 0$. Then (1) implies that $(b - a)c > 0$. Therefore $bc - ac > 0$. Finally by the additive property of order $bc > ac$. So, we have proved that $a < b$ and $c > 0$ imply $ac < bc$.

Case 2 Assume that $a < b$ and $c < 0$. Then by the additive property of order $-c > 0$ and by the previous item $a \cdot (-c) < b \cdot (-c)$. But $a \cdot (-c) = -(ac)$. Indeed, by consequence 1 of the distributive law (lecture 2) $-c = (-1)c$. Then one has the following chain of equalities

$$a \cdot (-c) = a \cdot ((-1) \cdot c) \overset{assoc.}{=} (a \cdot (-1)) \cdot c \overset{comm.}{=} ((-1) \cdot a)c \overset{assoc.}{=} (-1) \cdot (a \cdot c) = -ac$$

(in the last equality we used again consequence 1 of the distributive law from the lecture 2). In the same way $b \cdot (-c) = -(bc)$. So, $-(ac) < -(bc)$. Finally, using twice the additive property of order we get that $bc < ac$.