10.1: Sequences

A **sequence** is a list of numbers written in a definite order.

General sequence terms are denotes as follows:

$n=1$	a_1	–	*first*	*term*
$n=2$	a_2	–	*second*	*term*
		$\vdots$		
n	a_n	–	n^{th}	*term*
$n+1$	a_{n+1}	–	$(n+1)^{th}$	*term*
		$\vdots$		

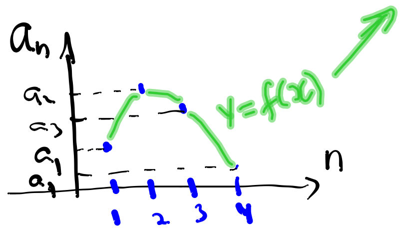
Notice that, in general, $a_{n+1} \neq a_n + 1$.

A sequence can be defined as a function whose domain is the set of positive whole numbers:

$$\mathbb{Z}_+ \longrightarrow \mathbb{R}$$

$$\{1, 2, 3, \dots\} \longrightarrow \{a_1, a_2, a_3, \dots\}$$

$$n \longrightarrow a_n = f(n)$$



NOTATION: $\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}, \quad \{a_n\}, \quad \{a_n\}_{n=1}^{\infty}.$

EXAMPLE 1. *Write down the first few terms of the following sequences:*

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{ \underset{n=1}{2}, \underset{n=2}{\frac{3}{4}}, \underset{n=3}{\frac{4}{9}}, \underset{n=4}{\frac{5}{16}}, \cdots \right\}$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{ -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \cdots \right\}$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

$f_1 = 1$

$f_2 = 1$

$f_3 = f_1 + f_2 = 1+1 = 2$

$f_4 = f_2 + f_3 = 1+2 = 3$

$f_5 = f_4 + f_3 = 3+2 = 5$

$\vdots$

$\boxed{1, 1, 2, 3, 5, 8, 13, 21, \ldots}$

EXAMPLE 2. *Find a general formula for the sequence:*

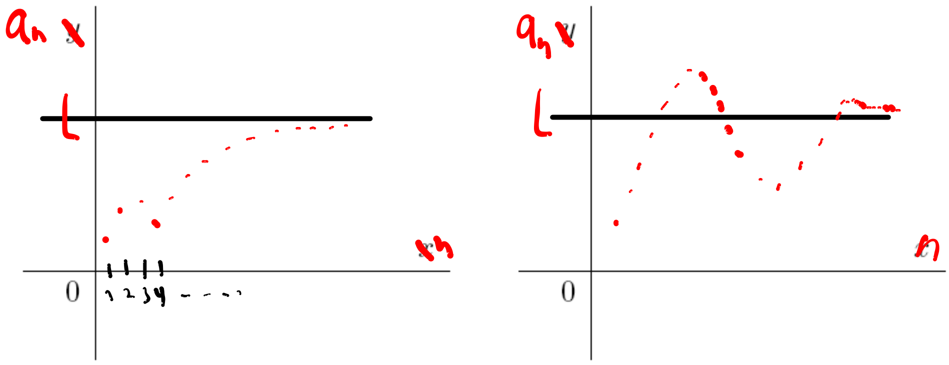
(a) $\left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \ldots\right\} = \left\{\frac{1}{2n+1}\right\}_{n=1}^{\infty}$

(b) $\left\{-\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots\right\} = \left\{\frac{(-1)^{n+1}}{n^2}\right\}_{n=2}^{\infty}$

$2^2 \quad 3^2 \quad 4^2 \quad 5^2$

DEFINITION 3. *If* $\lim_{n\to\infty} a_n$ *exists* and finite *then we say that the sequence* $\{a_n\}$ **converges** *(or is* **convergent.***) Otherwise, we say the sequence* $\{a_n\}$ **diverges** *(or is* **divergent.***)*

Graphs of two sequences with $\lim_{n\to\infty} a_n = L$.



EXAMPLE 4. *Determine if* $\{a_n\}_{n=1}^{\infty}$ *converges or diverges. If converges, find its limit.*

(a) $a_n = \dfrac{n+1}{2n+3}$

$$\lim_{n\to\infty} \frac{n+1}{2n+3} = \lim_{n\to\infty} \frac{\frac{n+1}{n}}{\frac{2n+3}{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2}$$

convergent

(b) $a_n = \dfrac{3n^2-1}{10n+5n^2}$

$$\lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2} = \frac{3}{5}$$

(As limit of rational function at ∞)

convergent

(c) $a_n = \arctan(2n)$

$$\lim_{n\to\infty} \arctan(2n) = \lim_{m\to\infty} \arctan(m) = \frac{\pi}{2}$$ convergent

(d) $a_n = \ln(2n+4) - \ln n = \ln \frac{2n+4}{n} = \ln\left(2 + \frac{4}{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(2 + \frac{4}{n}\right) = \ln\left(\lim_{n\to\infty}\left(2+\frac{4}{n}\right)\right)$$

$$= \boxed{\ln 2}$$ convergent

(e) $a_n = \cos \frac{\pi n}{2}$, $n \geq 1$

$a_1 = \cos \frac{\pi}{2} = 0$

$\{a_n\} = \{0, -1, 0, 1, 0, -1, 0, 1, \ldots\}$

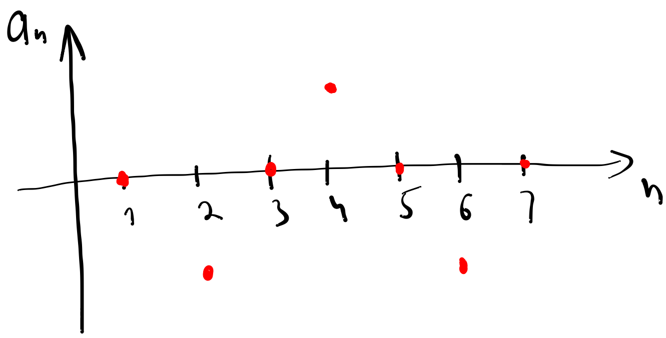
$a_2 = \cos \pi = -1$

$a_3 = \cos \frac{3\pi}{2} = 0$

$a_4 = \cos 2\pi = 1$

$a_5 = \cos \frac{5\pi}{2} = 0$

$\vdots$



Terms of the sequence oscilate between -1 and 1. Thus, the sequence doesn't approach any number. It is divergent.

Note: $b_n = \cos 2\pi n \Rightarrow b_n = 1$ for all n,

i.e. $\{b_n\}$ is constant sequence

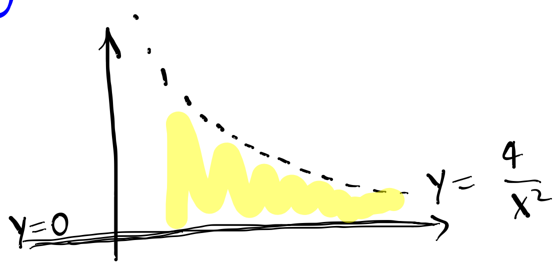
$\lim_{n\to\infty} b_n = 1 \Rightarrow \{b_n\}$ is convergent

(f) $a_n = \dfrac{3+(-1)^n}{n^2}$ $= \begin{cases} \dfrac{4}{n^2} & \text{for even } n \\ \dfrac{2}{n^2} & \text{for odd } n \end{cases}$

Squeeze Theorem

$$0 < a_n = \frac{3+(-1)^n}{n^2} \leq \frac{4}{n^2} \xrightarrow[n\to\infty]{} 0$$

$$\lim_{n\to\infty} a_n = 0$$



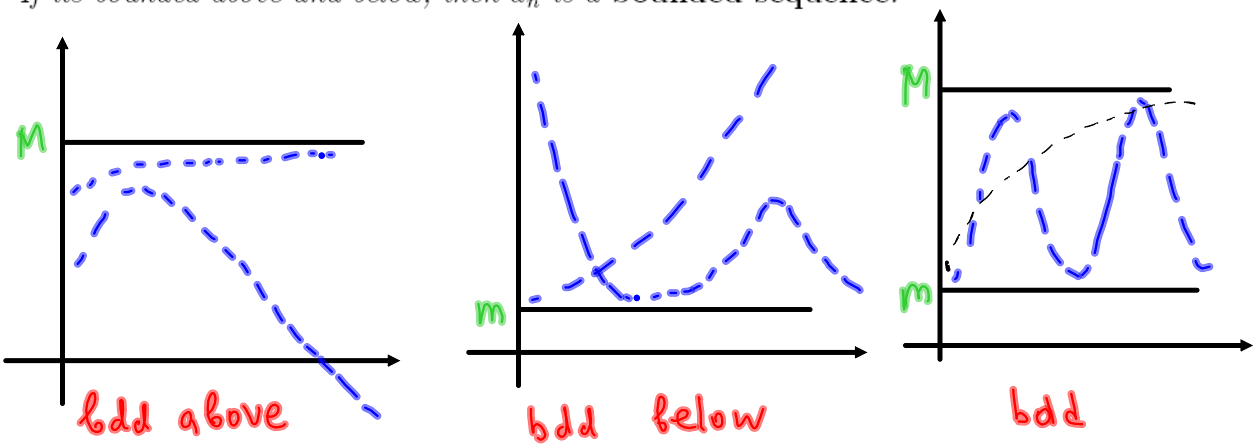
DEFINITION 5. *A sequence* $\{a_n\}$ *is* **bounded above** *if there is a number* M *s.t.*

$$a_n \leq M \quad \textit{for all} \quad n.$$

A sequence $\{a_n\}$ *is* **bounded below** *if there is a number* m *s.t.*

$$m \leq a_n \quad \textit{for all} \quad n.$$

If its bounded above and below, then a_n *is a* **bounded sequence**.



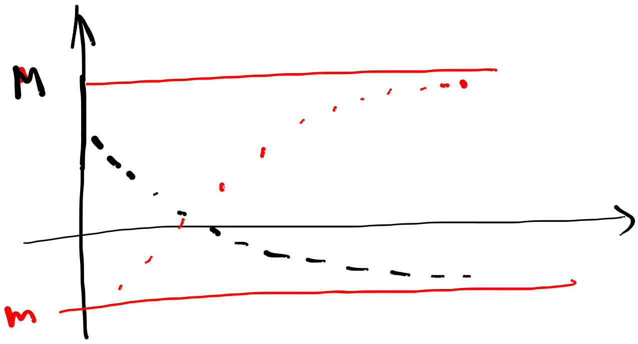
DEFINITION 6. *A sequence* $\{a_n\}$ *is* **increasing** *if*

$$\boxed{a_n < a_{n+1} \quad \text{for all} \quad n.}$$

A sequence $\{a_n\}$ *is* **decreasing** *if*

$$\boxed{a_n > a_{n+1} \quad \text{for all} \quad n.}$$

MONOTONIC SEQUENCE THEOREM. $\boxed{\textit{Every bounded, monotonic sequence is convergent.}}$



EXAMPLE 7. *Determine whether a_n is increasing, decreasing or not monotonic.*

(a) $a_n = -n^2$, $n \geq 1$

Way 1

$n < n+1$

$n^2 < (n+1)^2$

$-n^2 > -(n+1)^2 \Rightarrow a_n > a_{n+1} \Rightarrow \{a_n\}$ is decreasing

Way 2 $\quad a_n = -n^2 = f(n)$, $n \geq 1$

$f(x) = -x^2$, $x \geq 1$

$f'(x) = -2x < 0$ for $x \geq 1$

$f \downarrow \Rightarrow \{a_n\}$ is decreasing

(b) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$

$f(x) = \dfrac{2}{x^2}$, $x \geq 5$

$f(n) = \dfrac{2}{n^2}$

$f'(x) = \left(\dfrac{2}{x^2}\right)' = -\dfrac{4}{x^3} < 0 \Rightarrow f \downarrow \Rightarrow \left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$ is decreas.

(c) $\{(-1)^{n+1}\}_{n=1}^{\infty}$ $= \{1, -1, 1, -1, 1, -1 \dots\}$ oscilates

not monotonic

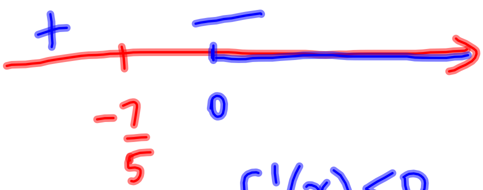
(d) $a_n = \dfrac{\sqrt{n+1}}{5n+3}$, $n = 0, 1, 2 \dots$

Define $f(x) = \dfrac{\sqrt{x+1}}{5x+3}$ so that for all $x \geq 0$ $f(n) = a_n$

$$f'(x) = \frac{\frac{1}{2\sqrt{x+1}}(5x+3) - 5\sqrt{x+1}}{(5x+3)^2} = \text{common denom.}$$

$$f'(x) = \frac{5x+3 - 10(x+1)}{2\sqrt{x+1}\,(5x+3)^2} = -\frac{5x+7}{2\sqrt{x+1}\,(5x+3)^2}$$

>0



$f'(x) < 0$ for all $x \geq 0$

$\Rightarrow f(n) \downarrow$ as $n \to \infty$

$\{a_n\}_{n=0}^{\infty}$ is decreasing

Example 8.

Consider the sequence defined by $a_1 = 1, \quad a_{n+1} = \dfrac{1}{3-a_n}$

Find the first 5 terms of this sequence. Find the limit of the sequence.

$$1, \ \frac{1}{2}, \ \frac{2}{5}, \ \frac{5}{13}, \ \frac{13}{34}, $$

$a_1 = 1$

$a_2 = \dfrac{1}{3-a_1} = \dfrac{1}{2}$

$a_3 = \dfrac{1}{3-\frac{1}{2}} = \dfrac{2}{5}$

$a_4 = \dfrac{1}{3-\frac{2}{5}} = \dfrac{5}{13}$

$a_5 = \dfrac{1}{3-5/13} = \dfrac{13}{34}$

$\{a_n\}$ is bounded

$0 < a_n < 1$

decreasing (monotonic)

$\Rightarrow$

$\Rightarrow$ convergent

There exist a real number L such that

$$L = \lim_{n\to\infty} a_n$$

Note here that $0 \le L < 1$

Find L.

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n} = \frac{1}{3-L}$$

because $a_n \neq 3$

$$L = \frac{1}{3-L} \Rightarrow L(3-L) = 1$$

$$3L - L^2 = 1$$

$$L^2 - 3L + 1 = 0$$

$$L_{1,2} = \frac{3 \pm \sqrt{3^2 - 4\cdot 1\cdot 1}}{2}$$

$$L_{1,2} = \frac{3 \pm \sqrt{5}}{2} \quad b/c \quad 0 \le L < 1$$

$$L = \boxed{\frac{3-\sqrt{5}}{2}}$$