

10.6: Representation of Functions as Power Series

Problem: Find the sum of the following series: $\sum_{n=0}^{\infty} x^n$

$\frac{1}{1-x}, |x| < 1$

power series
centered at 0, $C_n = 1$
with $R = 1$
interval of conv. $-1 < x < 1$, or $(-1, 1)$

geometric series
 $a = 1, r = x$
Converges when $|r| < 1$,
or $|x| < 1$

EXAMPLE 1. Find a power series representation for $f(x)$ and determine the interval of convergence.

(a) $f(x) = \frac{1}{5-x} = \frac{1}{5} \cdot \frac{1}{1-\frac{x}{5}} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{1}{5} \frac{x^n}{5^n}$

Sum of geom. series

$$\frac{1}{5-x} = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$$

$$\left|\frac{x}{5}\right| < 1$$

$$|x| < 5$$

$$\Rightarrow (-5, 5)$$

(b) $f(x) = \frac{x}{5-x} = x \frac{1}{5-x} = x \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$

$$(-5, 5)$$

$$\frac{x}{5-x} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}$$

$$(c) f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n, \text{ where } |-x^2| < 1$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\Downarrow$$
$$|x^2| < 1$$

$$|x| < 1$$

$$\boxed{(-1, 1)}$$

$$(d) f(x) = \frac{x^2}{16x^4 - 9} = x^2 \cdot \frac{1}{-(9 - 16x^4)} = -\frac{x^2}{9} \cdot \frac{1}{1 - \frac{16x^4}{9}}$$

$$f(x) = -\frac{x^2}{9} \cdot \sum_{n=0}^{\infty} \left(\frac{16x^4}{9}\right)^n, \quad \left|\frac{16x^4}{9}\right| < 1$$

Sum of geom. series

$$f(x) = \sum_{h=0}^{\infty} -\frac{x^2}{9} \cdot \frac{16^n}{9^n} x^{4n}$$

$$x^4 < \frac{9}{16}$$

$$x^2 < \frac{3}{4}$$

$$|x| < \frac{\sqrt{3}}{2}$$

$$f(x) = \sum_{h=0}^{\infty} -\frac{16^n}{9^{n+1}} x^{4n+2}$$

$$\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$$

Differentiation and Integration of power series

THEOREM 2. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

sum of power series $\leftarrow f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

- $f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$ **TERMWISE DIFFERENTIATION**
- $\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$ **TERMWISE INTEGRATION**

The radii of convergence of the power series for $f'(x)$ and $\int f(x) dx$ are both R .

$$f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx} (x-a)^n$$

$$= \sum_{n=0}^{\infty} c_n \cdot n (x-a)^{n-1}$$

EXAMPLE 3. Find a power series representation for $f(x)$ and determine the radius of convergence.

(a) $f(x) = \frac{1}{(5-x)^2}$

Integrate w.r.t. x :

$$\int f(x) dx = \int \frac{dx}{(5-x)^2} = \frac{1}{5-x} + C \quad \text{By Ex. 1a}$$

u-subst.

$$= \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} + C, \text{ where } |x| < 5$$

$R = 5$

Return to $f(x)$:

$$\frac{d}{dx} \int f(x) dx = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} + C$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{5^{n+1}} \frac{d}{dx} (x^n) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{5^{n+1}}, \quad R = 5$$

(b) $f(x) = \ln(2+x)$

$$f'(x) = \frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1+\frac{x}{2}} = \frac{1}{2} \cdot \frac{1}{1 - (-\frac{x}{2})}$$

Sum of geom. series

$$f'(x) = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

where $|\frac{-x}{2}| < 1$

$$f'(x) = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{(-1)^n x^n}{2^n}$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

Integrate

$$\int f'(x) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \int x^n dx$$

$$\ln(2+x) = f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{n+1} (n+1)} + C, \quad R=2$$

To determine C we can plugin a point from the interval of convergence

$$\ln(2+x) = \boxed{} x + \boxed{} x^2 + \boxed{} x^3 + \dots + C$$

$$x=0 \Rightarrow \ln 2 = 0 + C \Rightarrow \boxed{C = \ln 2}$$

$$\ln(2+x) = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{n+1} (n+1)}, \quad R=2$$

(c) $f(x) = \arctan(2x)$

$$f'(x) = \frac{d}{dx}(\arctan(2x)) \stackrel{\text{Chain Rule}}{=} \frac{2}{1+(2x)^2} = 2 \cdot \frac{1}{1+4x^2}$$

$$f'(x) = 2 \cdot \frac{1}{1-(-4x^2)} = 2 \sum_{n=0}^{\infty} (-4x^2)^n \quad \text{where } |4x^2| < 1$$

$x^2 < \frac{1}{4}$
 $|x| < \frac{1}{2}$

$$f'(x) = \sum_{n=0}^{\infty} 2 \cdot (-1)^n (2^2)^n (x^2)^n$$

$$\boxed{R = \frac{1}{2}}$$

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n}, \quad R = \frac{1}{2}$$

Integrate: $\int f'(x) dx = \sum_{n=0}^{\infty} (-1)^n \cdot 2^{2n+1} \int x^{2n} dx$

$$\arctan(2x) = f(x) = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{x^{2n+1}}{2n+1} + C, \quad R = \frac{1}{2}$$

To find C plug in a point from interval of convergence (namely, the center of the series, $x=0$)

$$\arctan(2 \cdot 0) = 0 + C \Rightarrow C = 0$$

$$\boxed{\arctan 2x = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{x^{2n+1}}{2n+1}, \quad R = \frac{1}{2}}$$

EXAMPLE 4. Use a power series decomposition to approximate the integral

Means that we need $|R_n| \sim 10^{-7}$ or less to six decimal places.

$$\int_0^{0.2} \frac{1}{1+x^4} dx$$

sum of geom. series $r = -x^4$

$|x| < 1$

$$\int_0^{0.2} \frac{dx}{1+x^4} = \int_0^{1/5} \sum_{n=0}^{\infty} (-x^4)^n dx = \int_0^{1/5} \sum_{n=0}^{\infty} (-1)^n x^{4n} dx = \sum_{n=0}^{\infty} (-1)^n \int_0^{1/5} x^{4n} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1} \Big|_0^{1/5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} \cdot \frac{1}{5^{4n+1}}$$

Alternating series

$$\int_0^{1/5} \frac{dx}{1+x^4} = \frac{1}{5} - \frac{1}{5 \cdot 5^5} + \frac{1}{9 \cdot 5^9} - \frac{1}{13 \cdot 5^{13}} + \dots$$

$$S \approx S_0 \pm R_0$$

$$S \approx 0.2 \pm 6.4 \cdot 10^{-5}$$

$$S \approx S_1 \pm R_1$$

$$S_1 = \frac{1}{5} - \frac{1}{5 \cdot 5^5} \approx 0.199936$$

$$R_0 = \frac{1}{5 \cdot 5^2} \approx 6.4 \cdot 10^{-5}$$

$$|R_1| \leq \frac{1}{9 \cdot 5^9} \approx 5.7 \cdot 10^{-8}$$