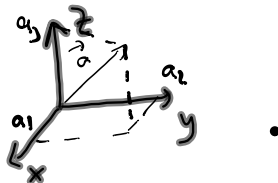


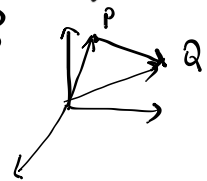
## 11.2: Vectors and the Dot Product in Three Dimensions

DEFINITION 1. A 3-dimensional vector is an ordered triple  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$



Given the points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\overrightarrow{PQ}$  is

$$\overrightarrow{PQ} = \mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \overrightarrow{OQ} - \overrightarrow{OP}$$

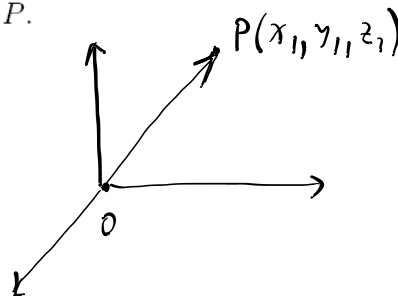


The representation of the vector that starts at the point  $O(0,0,0)$  and ends at the point  $P(x_1, y_1, z_1)$  is called the **position vector** of the point  $P$ .

$$\overrightarrow{OP} = \langle x_1, y_1, z_1 \rangle$$

"

$$\langle x_1 - 0, y_1 - 0, z_1 - 0 \rangle$$



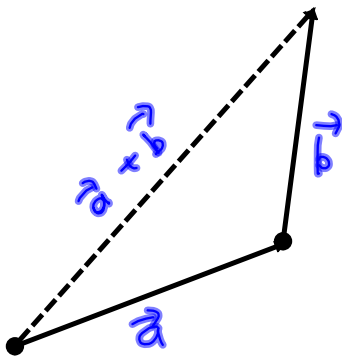
EXAMPLE 2. Find the vector represented by the directed line segment with the initial point  $A(1, 2, 3)$  and terminal point  $B(3, 2, -1)$ . What is the position vector of the point  $A$ ?  $= \vec{OA} = \langle 1, 2, 3 \rangle$

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} = \langle 3, 2, -1 \rangle - \langle 1, 2, 3 \rangle \\ &= \langle 3-1, 2-2, -1-3 \rangle = \langle 2, 0, -4 \rangle\end{aligned}$$

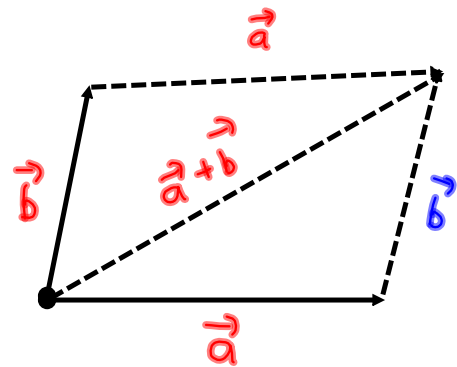
**Vector Arithmetic:** Let  $a = \langle a_1, a_2, a_3 \rangle$  and  $b = \langle b_1, b_2, b_3 \rangle$ .

- Scalar Multiplication:  $\alpha a = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$ ,  $\alpha \in \mathbb{R}$ .
- Addition:  $a + b = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

TRIANGLE LAW



PARALLELOGRAM LAW



Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if one is a scalar multiple of the other, i.e. there exists  $\alpha \in \mathbb{R}$  s.t.  $\mathbf{b} = \alpha \mathbf{a}$ . Equivalently:

$$\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \vec{\mathbf{b}} = \alpha \vec{\mathbf{a}}$$

$$\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle \quad \vec{\mathbf{b}} = \langle b_1, b_2, b_3 \rangle$$

$$\langle b_1, b_2, b_3 \rangle = \alpha \langle a_1, a_2, a_3 \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$$

$$b_1 = \alpha a_1$$

$$b_2 = \alpha a_2 \Rightarrow$$

$$b_3 = \alpha a_3$$

if  $a_1 a_2 a_3 \neq 0$   
 $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$

For example,  $\vec{\mathbf{a}} = \langle 1, 2, 3 \rangle$ ,  $\vec{\mathbf{b}} = \langle -\frac{1}{2}, -1, -\frac{3}{2} \rangle$  }  $\Rightarrow$

Then  $\frac{1}{-\frac{1}{2}} = \frac{2}{-1} = \frac{3}{-\frac{3}{2}} = -2$

$$\Rightarrow \vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$$

$$\vec{\mathbf{c}} = \langle 1, 0, 3 \rangle \Rightarrow \vec{\mathbf{a}} \nparallel \vec{\mathbf{c}}$$

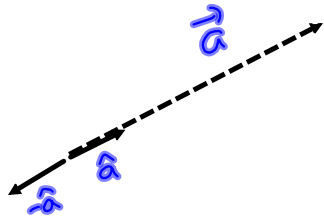
The magnitude or length of  $a = \langle a_1, a_2, a_3 \rangle$ :

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Zero vector:  $\mathbf{0} = \langle 0, 0, 0 \rangle$ ,  $|\mathbf{0}| = 0$ .

Note that  $|a| = 0 \Leftrightarrow a = \mathbf{0}$ .

$$\text{Unit vector: } \hat{a} = \frac{a}{|a|} = \frac{\langle a_1, a_2, a_3 \rangle}{\sqrt{a_1^2 + a_2^2 + a_3^2}} = \left\langle \frac{a_1}{\sqrt{\quad}}, \frac{a_2}{\sqrt{\quad}}, \frac{a_3}{\sqrt{\quad}} \right\rangle$$



$$|\hat{a}| = 1$$

Standard Basis Vectors:

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$

$$\mathbf{j} = \langle 0, 1, 0 \rangle$$

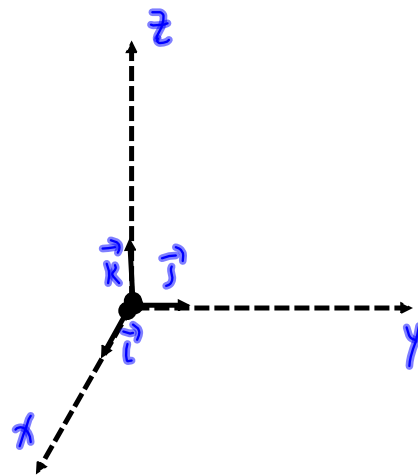
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

Note that  $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$ .

We have:

$$\begin{aligned} \mathbf{a} = \langle a_1, a_2, a_3 \rangle &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\ &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \end{aligned}$$

$$\langle 2, 3, -7 \rangle = 2\vec{i} + 3\vec{j} - 7\vec{k}$$



EXAMPLE 3. Given  $\mathbf{a} = \langle 1, 0, -3 \rangle$  and  $\mathbf{b} = \langle 3, 1, 2 \rangle$ . Find

(a)  $|\mathbf{b} - \mathbf{a}|$ .

$$\begin{aligned} \vec{\mathbf{b}} - \vec{\mathbf{a}} &= \langle 3, 1, 2 \rangle - \langle 1, 0, -3 \rangle = \langle 3-1, 1-0, 2-(-3) \rangle \\ |\vec{\mathbf{b}} - \vec{\mathbf{a}}| &= |\langle 2, 1, 5 \rangle| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{4 + 1 + 25} = \sqrt{30} \end{aligned}$$

(b) a unit vector that has the same direction as  $\mathbf{b}$ .

$$\hat{\mathbf{b}} = \frac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}$$

$$|\vec{\mathbf{b}}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$$

$$\hat{\mathbf{b}} = \frac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|} = \frac{\langle 3, 1, 2 \rangle}{\sqrt{14}} = \left\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle$$

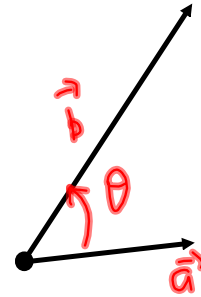
$$\text{OR } \hat{\mathbf{b}} = \frac{3}{\sqrt{14}} \hat{\mathbf{i}} + \frac{1}{\sqrt{14}} \hat{\mathbf{j}} + \frac{2}{\sqrt{14}} \hat{\mathbf{k}}$$

Dot Product of two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  is the NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ ,  $0 \leq \theta \leq \pi$ .

If  $\mathbf{a} = \mathbf{0}$  or  $\mathbf{b} = \mathbf{0}$  then  $\mathbf{a} \cdot \mathbf{b} = 0$ .



The same

Component Formula for dot product of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ :

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

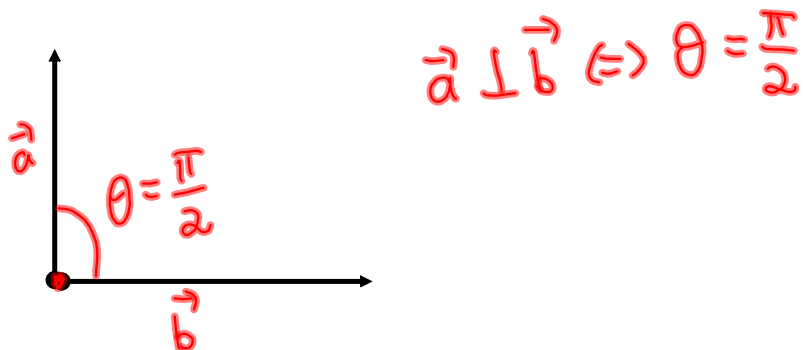
If  $\theta$  is the *angle* between two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|}$$

$$= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$$



DEFINITION 4. Two **nonzero** vectors  $\vec{a}$  and  $\vec{b}$  are called **perpendicular** or **orthogonal** if the angle between them is  $\theta = \pi/2$ .



$$\vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos \theta = |\vec{a}| \cdot |\vec{b}| \cdot \underbrace{\cos \frac{\pi}{2}}_0 = 0$$

If  $\vec{a} \cdot \vec{b} = 0 \Rightarrow \underbrace{|\vec{a}|}_{\neq 0} \cdot \underbrace{|\vec{b}|}_{\neq 0} \cos \theta = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Conclusion 
 $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$

EXAMPLE 6. For what value(s) of  $c$  are the vectors  $\underbrace{ci + 2j + k}_{\vec{a}}$  and  $\underbrace{4i + 3j + ck}_{\vec{b}}$  orthogonal?

$$\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$$

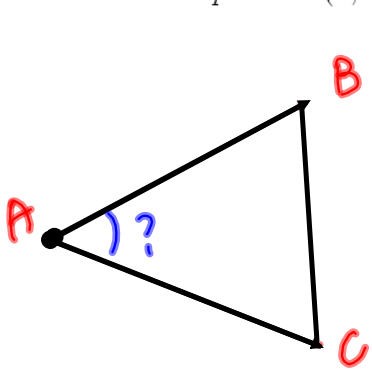
$$\langle c, 2, 1 \rangle \cdot \langle 4, 3, c \rangle = 0$$

$$4c + 2 \cdot 3 + 1c = 0$$

$$5c + 6 = 0$$

$$c = \boxed{-\frac{6}{5}}$$

EXAMPLE 7. The points  $A(6, -1, 0)$ ,  $B(-3, 1, 2)$ ,  $C(2, 4, 5)$  form a triangle. Find angle at  $A$ .



$$\angle A = \angle BAC = \angle \vec{AB}, \vec{AC}$$

$$\cos \angle A = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| \cdot |\vec{AC}|}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = \langle -3, 1, 2 \rangle - \langle 6, -1, 0 \rangle = \langle -3-6, 1-(-1), 2-0 \rangle$$

$$\vec{AB} = \langle -9, 2, 2 \rangle$$

$$\vec{AC} = \langle 2, 4, 5 \rangle - \langle 6, -1, 0 \rangle = \langle 2-6, 4-(-1), 5-0 \rangle = \langle -4, 5, 5 \rangle$$

$$\vec{AB} \cdot \vec{AC} = \langle -9, 2, 2 \rangle \cdot \langle -4, 5, 5 \rangle = -9 \cdot (-4) + 2 \cdot 5 + 2 \cdot 5 = 56$$

$$|\vec{AB}| = |\langle -9, 2, 2 \rangle| = \sqrt{(-9)^2 + 2^2 + 2^2} = \sqrt{81 + 4 + 4} = \sqrt{89}$$

$$|\vec{AC}| = |\langle -4, 5, 5 \rangle| = \sqrt{4^2 + 5^2 + 5^2} = \sqrt{66}$$

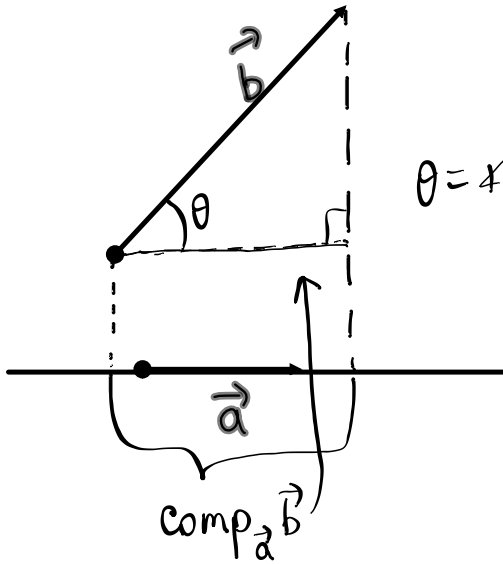
Plug in :

$$\cos \angle A = \frac{56}{\sqrt{89} \cdot \sqrt{66}}$$

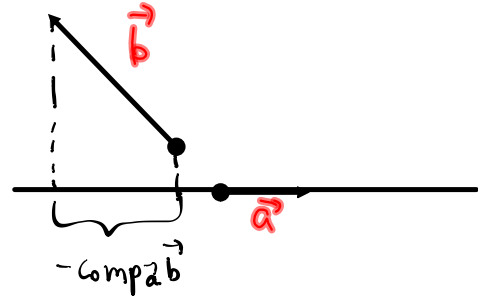
$$\angle A = \arccos \frac{56}{\sqrt{89 \cdot 66}} \approx 43^\circ$$

Projections:

- Scalar projection of vector  $\mathbf{b}$  onto vector  $\mathbf{a}$ :  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$



$$\theta = \angle \vec{a}, \vec{b}$$

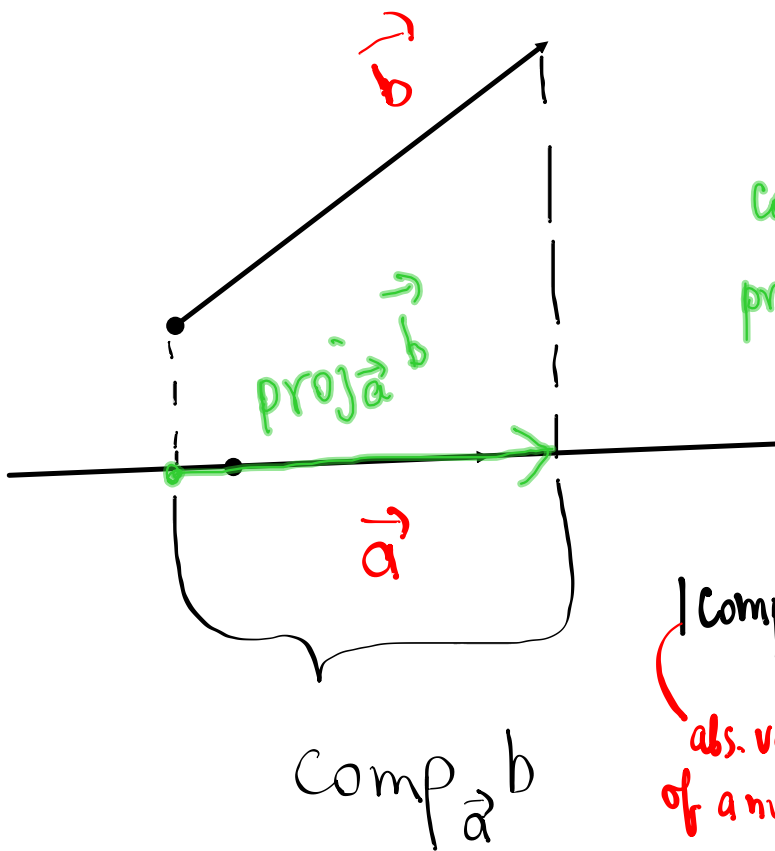


$$\text{comp}_{\mathbf{a}} \mathbf{b} = |\mathbf{b}| \cos \theta = \cancel{|\mathbf{b}|} \cdot \frac{\vec{a} \cdot \vec{b}}{\cancel{|\mathbf{a}|} \cdot \cancel{|\mathbf{b}|}} = \frac{\vec{a} \cdot \vec{b}}{|\mathbf{a}|}$$

- Vector projection of vector b onto vector a:

$$\text{proj}_{\vec{a}} \vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \frac{\vec{a}}{|\vec{a}|} = \frac{(\vec{a} \cdot \vec{b}) \vec{a}}{|\vec{a}|^2}$$

$\parallel$   
 $(\text{comp}_{\vec{a}} \vec{b}) \cdot \hat{\vec{a}}$



Note  
 $\text{comp}_{\vec{a}} \vec{b}$  is a number  
 $\text{proj}_{\vec{a}} \vec{b}$  is a vector

$$|\text{comp}_{\vec{a}} \vec{b}| = |\text{proj}_{\vec{a}} \vec{b}|$$

abs. value of a number      magnitude of a vector

EXAMPLE 8. Find the scalar and vector projections of  $\langle 2, -2, -1 \rangle$  onto  $\langle 3, 3, 4 \rangle$ .

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{\langle 2, -2, -1 \rangle \cdot \langle 3, 3, 4 \rangle}{|\langle 3, 3, 4 \rangle|}$$

$$= \frac{6 - 6 - 4}{\sqrt{9 + 9 + 16}} = -\frac{4}{\sqrt{34}}$$

$$\text{proj}_{\vec{v}} \vec{u} = \text{comp}_{\vec{v}} \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} = -\frac{4}{\sqrt{34}} \cdot \frac{\langle 3, 3, 4 \rangle}{\sqrt{34}}$$

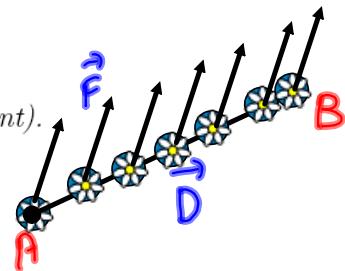
$$= -\frac{4}{34} \langle 3, 3, 4 \rangle = -\frac{2}{17} \langle 3, 3, 4 \rangle$$

$$\text{proj}_{\vec{v}} \vec{u} = \left\langle -\frac{6}{17}, -\frac{6}{17}, -\frac{8}{17} \right\rangle$$

DEFINITION 8. The **work** done by a force  $\mathbf{F}$  in moving an object from point  $A$  to point  $B$  is given by

$$W = \mathbf{F} \cdot \mathbf{D}$$

where  $\mathbf{D} = \overrightarrow{AB}$  is the distance the object has moved (or displacement).



EXAMPLE 9. A force is given by a vector  $\mathbf{F} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$  and moves a particle from the point  $P(1, 2, 0)$  to the point  $Q(2, 3, 5)$ . Find the work done.

$$\vec{D} = \vec{PQ} = \vec{OQ} - \vec{OP} = \langle 2, 3, 5 \rangle - \langle 1, 2, 0 \rangle = \langle 1, 1, 5 \rangle$$

$$W = \vec{F} \cdot \vec{D} = \langle 1, -1, 5 \rangle \cdot \langle 1, 1, 5 \rangle = 1 - 1 + 25 = \boxed{25}$$