Homework 4 Solution MATH 666, Fall 11

In our case $\dot{x} = Ax + bu$, $x \in \mathbb{R}^2$ where

$$A = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \qquad (1)$$

Check General Position Condition (GPC) (not necessary to do it)

$$Ab = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 5-16 \\ 5-12 \end{pmatrix} = \begin{pmatrix} -9 \\ -7 \end{pmatrix}$$

$$\det(b, Ab) = \begin{vmatrix} -5 & -9 \\ -4 & -7 \end{vmatrix} = 35-36 = -1 \neq 0 \Rightarrow \text{GPC holds} \Rightarrow \text{extremal}$$
controls are bang-bang

Find the spectrum of A.

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -4 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - \operatorname{tr} A\,\lambda + \det A = \lambda^2 - 2\lambda + 1 = 0$$

$\lambda = 1 \Rightarrow$ we have the unique eigenvalue $\lambda = 1$ (and the Jordan normal form of A is $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$)

In particular, $\operatorname{spec} A \subset \mathbb{R} \Rightarrow$ an extremal control function has no more than $\underset{n=2}{}1$ switch

Indeed, $h(p, x, u) = p_1(-x_1 + 4x_2) + p_2(-x_1 + 3x_2) + (-5p_1 - 4p_2)u$

So from the maximality conditions: $u(t) = 2$ if $-5p_1(t) + 4p_2(t) < 0$ and $u(t) = -1$ if $5p_1(t) + 4p_2(t) > 0$.

The covector P satisfies the adjoint equation

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = p_1 + p_2$$
$$\iff \dot{p} = -A^T p$$
$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -4p_1 - 3p_2$$

since spec $(-A^T) = \{-1\}$ and A is not multiple of I

$5p_1(t) + 4p_2(t)$ is a quasi-polynomial of the form

$$(c_1 + c_2 t) e^{-t}$$

This quasipolynomial has no more than one zero $\Rightarrow$
$u(t)$ is bang-bang and has no more than one switch.

An extremal trajectories are either a trajectory of system 1 with $u \equiv 2$ or $u \equiv -1$ or a concatenation of two such trajectories.

Describe the phase portrait of system (1) with $u = 2$ and $u = -1$ separately

1) $u = 2$

$$\begin{cases} \dot{x}_1 = -x_1 + 4x_2 - 10 \\ \dot{x}_2 = -x_1 + 3x_2 - 8 \end{cases} \quad (2)$$

The stationary point satisfies: $\begin{cases} -x_1 + 4x_2 - 10 = 0 \\ -x_1 + 3x_2 - 8 = 0 \end{cases} \Rightarrow$ $x_2 - 2 = 0 \Rightarrow x_2 = 2 \Rightarrow$ $x_1 = 4x_2 - 10 = -2$

so, it is $(-2, 2)$

Since spec $A = \{1\}$ and $A \neq I$, the point $(-2, 2)$ is a degenerate (improper) unstable node.

An eigenvector $\begin{pmatrix}v_1\\v_2\end{pmatrix}$ of $\lambda=1$ satisfies

$$(A-I)v=\begin{pmatrix}-2&4\\-1&2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$ $\Rightarrow$ one can take $\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$ $\Rightarrow$

the direction of the tangent lines to all trajectories converge to the direction of $\pm\begin{pmatrix}2\\1\end{pmatrix}$ when $t\to-\infty$

Find a generalized eigenvector w solving (for figuring out the phase portrait)

$$(A-I)w=v \Leftrightarrow \begin{pmatrix}-2&4\\-1&2\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$ $\Rightarrow$ as w one can

take $w=\begin{pmatrix}-1\\0\end{pmatrix}$ $\Rightarrow$ the general solution of (2) is

$$C_1e^t\begin{pmatrix}2\\1\end{pmatrix}+C_2\left(te^t\begin{pmatrix}2\\1\end{pmatrix}+e^t\begin{pmatrix}-1\\0\end{pmatrix}\right)+\begin{pmatrix}-2\\2\end{pmatrix}$$ for

arbitrary constants C_1 and C_2.

The phase portrait of (2) is

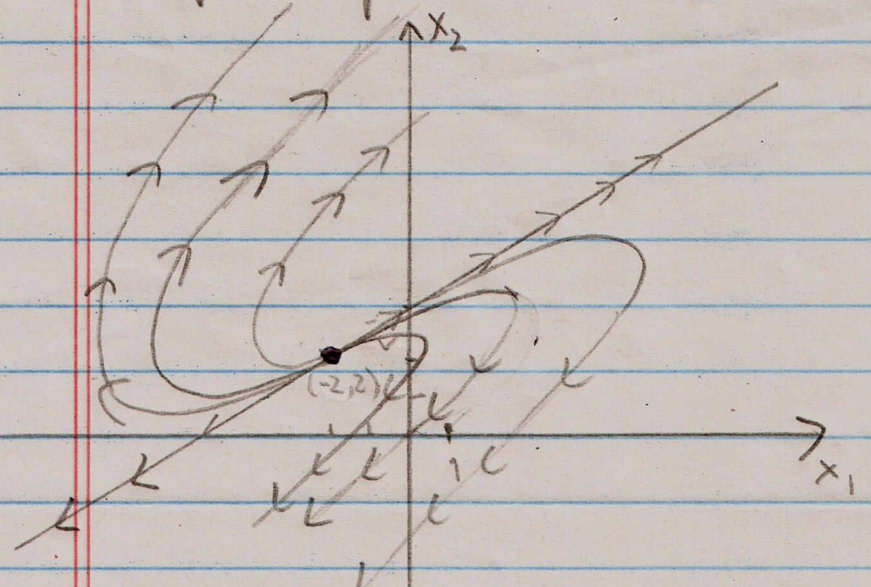


Fig. 1

2) $u = -1$

$$\begin{cases} \dot{x}_1 = -x_1 + 4x_2 + 5 \\ \dot{x}_2 = -x_1 + 3x_2 + 4 \end{cases} \qquad (3)$$

The stationary point satisfies: $\begin{cases} -x_1 + 4x_2 + 5 = 0 \\ -x_1 + 3x_2 + 4 = 0 \end{cases} \Rightarrow x_2 = -1 \Rightarrow x_1 = 1 \Rightarrow$ the stationary points $(1,-1)$.

Then using the previous item the general solution of (3) is

$$C_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \left(t e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for}$$

arbitrary constants C_1 and C_2.

The phase portrait of (3) is:

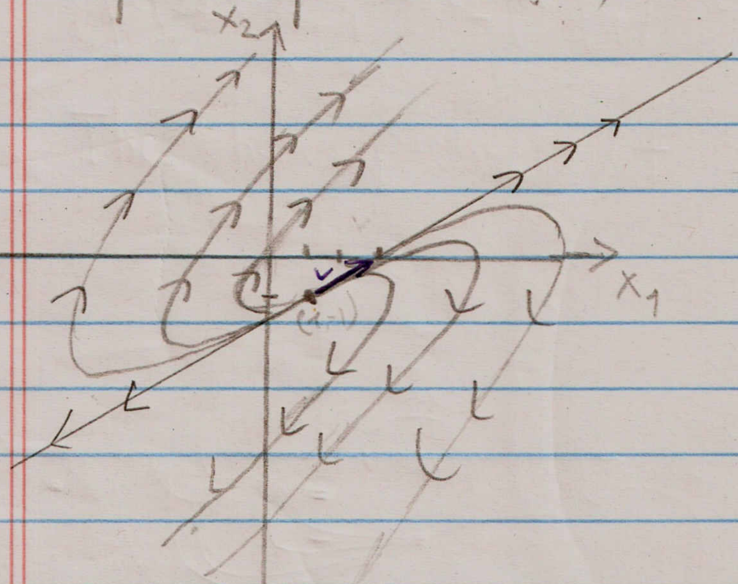


Fig. 2

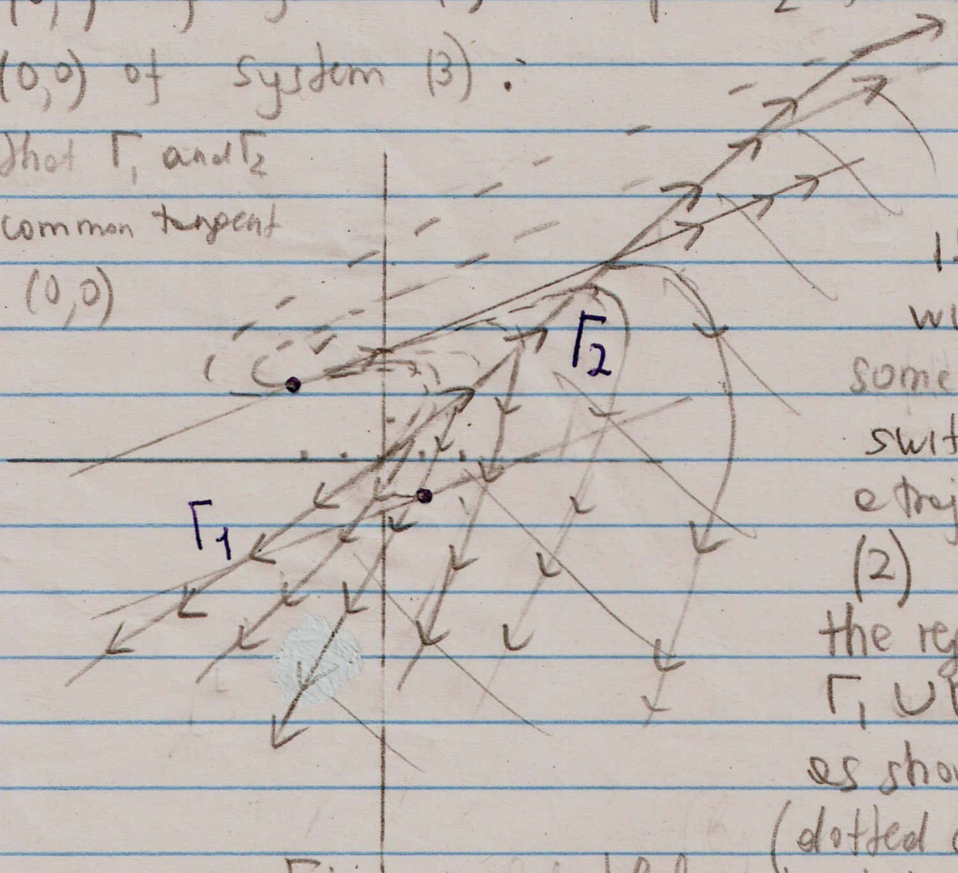
3) Attainable set from the origin

By Theorem on existence of optimal control steering from x_0 to x_1 if there exists at least one admissible trajectory from x_0 to x_1 it follows that

$A_{(0,0)}$ = the set of all points that can be reached from $(0,0)$ by extremal trajectories.

The starting piece of an extremal trajectory from $(0,0)$ is either a piece Γ_1 of the trajectory from $(0,0)$ of system (2) or a piece Γ_2 of the trajectory from $(0,0)$ of system (3).

Note that Γ_1 and Γ_2 have the common tangent line at $(0,0)$



If one starts with Γ_2 then after some time one may switch to a trajectory of system (2) and this way the region Below $\Gamma_1 \cup \Gamma_2$ is filled as shown on Fig. 3

Fig. 3

(dotted curves here do not belong to extremal trajectories, it is drawn just to show that the second part of the optimal traj. comes from system (2))

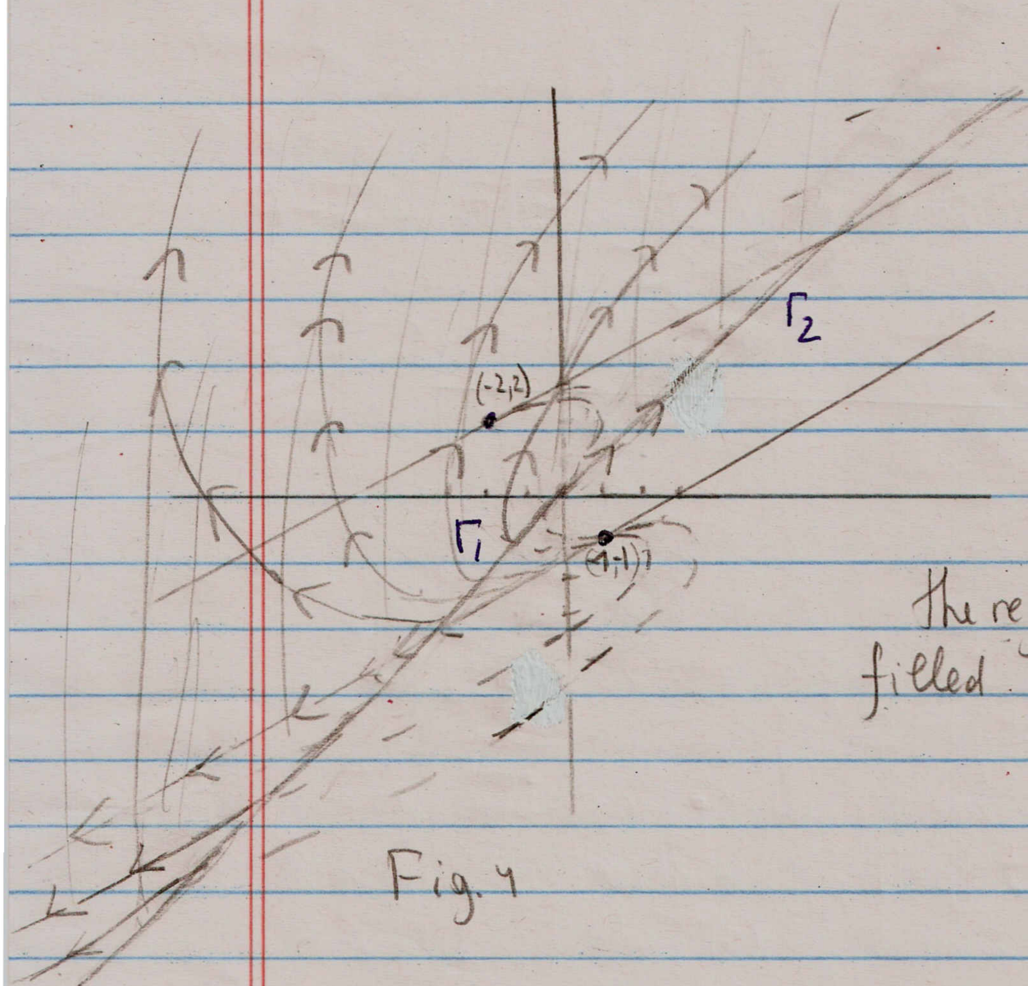


Fig. 4

If one starts with Γ_1 then after some time one may switch to a trajectory of system (3) and in this way the region above $\Gamma_1 \cup \Gamma_2$ is filled as shown on Fig. 4

Conclusion $\mathcal{A}_{(0,0)} = \mathbb{R}^2$

Rem In general, if spec A is in the right half-plane (i.e. A is unstable) and 0 is an inner point of the polytope U, then the attainable set of the origin $= R^n$ (see Theorem 14 of the book of Pontryagin and others).

4) Optimal synthesis with the target (0,0)

In this case the final piece of an optimal trajectory is either a piece $\tilde{\Gamma}_1$ of the trajectory of system (2) entering (0,0) or a piece $\tilde{\Gamma}_2$ of the trajectory of system (3) entering (0,0)

Then $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$ is the switching curve for the optimal control

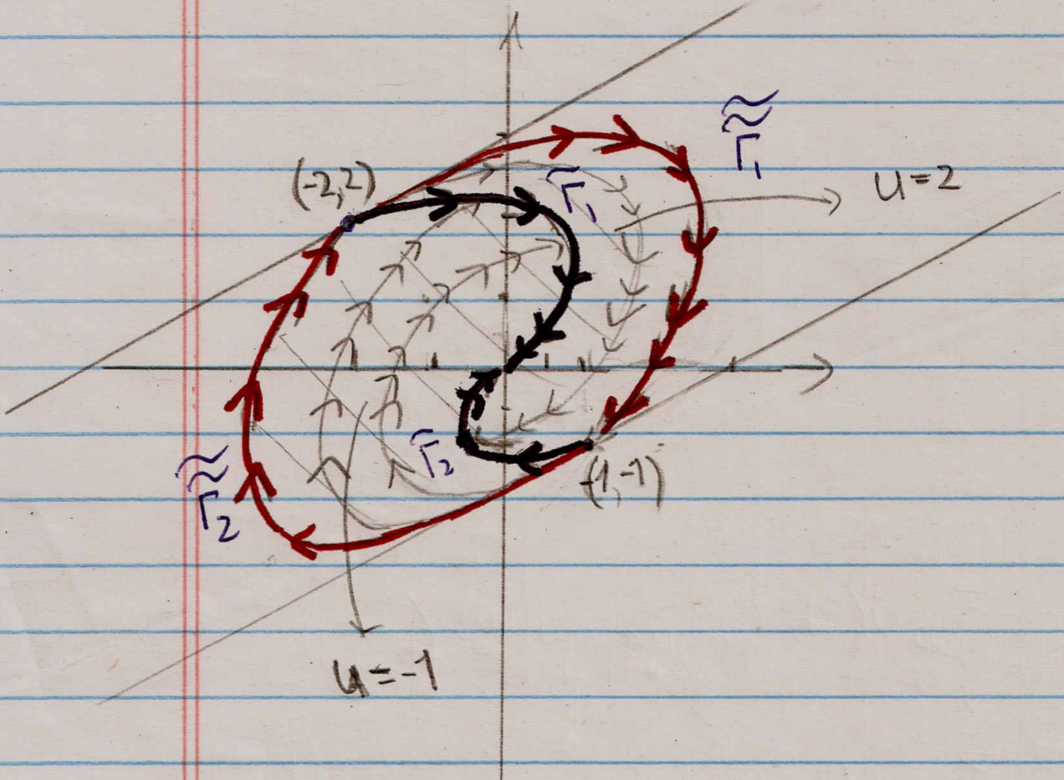


Fig 5

Let $\tilde{\tilde{\Gamma}}_1$ be the piece of the trajectory of system (2) entering $(1,-1)$ (the stationary point of system (3)) and $\tilde{\tilde{\Gamma}}_2$ be the piece of the trajectory of system (3) entering $(-2,2)$ (the stationary point of system (2)). Then by constructions the target $(0,0)$ can be reached only from points of the region between (not including the boundaries) $\tilde{\tilde{\Gamma}}_1$ and $\tilde{\tilde{\Gamma}}_2$ ($\tilde{\Gamma}_1$ cannot be reached by a trajectory of system (3) from a point out of this region and $\tilde{\Gamma}_2$ cannot be reached by a trajectory of system (2) out of this region)

The optimal synthesis

$$u(x) = \begin{cases} -1 & \text{if } x \text{ is between } \tilde{\tilde{\Gamma}}_2 \text{ and } \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \\ 2 & \text{if } x \text{ is between } \tilde{\tilde{\Gamma}}_1 \text{ and } \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \end{cases}$$