# Homework Assignment 3 in Geometric Control Theory, MATH666

due to Oct 21, 2011

**Problem 1** Consider the control of angular momentum $M$ of a rigid body with a fixed point by two torques in the direction of two axis of inertia. It is described by the following control system:

$$\dot{M} = M \times A^{-1}M + u_1 l_1 + u_2 l_2, \tag{1}$$

where $A$ is the inertia operator of the body, $l_1$ and $l_2$ are two torques parallel to the inertia axis $\mathbb{R}e_1$, $\mathbb{R}e_2$, respectively, and both controls $u_1$ and $u_2$ take values in the set $\{-1, 1\}$. Under what conditions on the principle moments of inertia , i.e. the eigenvalues of the inertia operator $A$, the system (1) is controllable? Prove your answer.

**Problem 2**

**a)** Let $M = SO(3)$, the group of all $3 \times 3$ orthogonal matrices with determinant equal to 1. Consider the following control system with the state space $M$:

$$\dot{E} = E \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & u \\ 0 & -u & 0 \end{pmatrix}, \tag{2}$$

where $E \in M$ and $u \in \{-1, 1\}$. Is this system controllable?

**b)** Investigate the same question if $u \in \{1, 2\}$ (i.e. if we replace the control space $\{-1, 1\}$ by $\{1, 2\}$).

**Remark 1.** (The geometric interpretation of Problem 2): Equation (2) is nothing but the equation for the moving Frenet frame for a curve in $\mathbb{R}^3$ with the curvature 1 and the torsion $u$ (the frame consist of the columns of the matrix $E$). Then the problem 2 can be reformulated as follows: given two orthonormal frames $E_0$ and $E_1$ (defining the same orientation in $\mathbb{R}^3$) can we find a concatenation of curves in $\mathbb{R}^3$ with the curvature 1 and the torsion 1 or $-1$ such that the Frenet frame in the initial point is equal to $E_0$ and the Frenet frame at the end point is equal to $E_1$ (we assume that at the time moments of switching of control, the Frenet frames is continuous). Shortly speaking we control the Frenet frame by controlling the torsion.