Geometry of rank 2 distributions via abnormal extremals: algebraic structure of invariants and absolute parallelism

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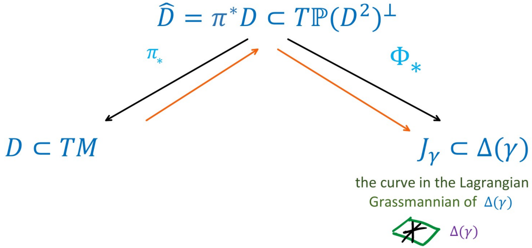
Grieg seminar, talk 2

June 15, 2021



The Jacobi curve J_γ produces the curve of flags in $\Delta(\gamma)$ via a series of osculations and skew-orthogonal complements:

$$\cdots \subset J_\gamma^{(-\nu)} \subset \cdots \subset J_\gamma^{(0)} = J_\gamma \subset J_\gamma^{(1)} \subset \cdots \subset J_\gamma^{(\nu)} \subset \cdots \subset \Delta(\gamma)$$

The curve $J_\gamma^{(4-n)}$ is called convex at λ if $\dim J_\gamma^{(4-n)} = 1$, i.e. near λ, $J_\gamma^{(4-n)}$ is a curve in the projective space $\mathbb{P}\Delta(\gamma)$ (moreover, it is the self-dual curve in the projective space)

Summary of geometry of curves in projective space

1. *Canonical projective structure on a curve*: i.e. the set of distinguished parametrizations defined up to a Möbius transformation.
2. If k is the dimension of the projective space, then for a convex curve in the projective space the set of fundamental invariants consists of $k-1$ relative invariants $\mathcal{W}_i$ of degree $i+2$, $i=1,\ldots,k-1$, called the Wilczynski invariants. Here $\mathcal{W}_i$ is a degree $i+2$ homogeneous polynomial on the tangent line at every point of the curve. In the given parametrization t it can be written as $\mathcal{W}_i(t)=A_i(t)\,dt^{i+2}$. The function $A_i(t)$ is called the density of the Wilczynski invariant w.r.t. the parameter t.
3. The curve in a projective space is self-dual if and only if all Wilczynski invariants of odd degree are equal to zero.

Summary on generalized Wilczynski invariants

Fix a local basis (X_1, X_2) of D. Set

$$X_3 := [X_1, X_2], \quad X_4 := [X_1, X_3], \quad X_5 := [X_2, X_3].$$

$\vec{h}_{X_1,X_2} := u_4 \overrightarrow{u}_2 - u_5 \overrightarrow{u}_1$, $\vec{h}_{X_1,X_2}$ is the generator of the characteristic foliation on $\mathbb{P}(D^2)^\perp$, associated with the local basis (X_1, X_2).

Let $\mathcal{R}_D \subset (D^2)^\perp \backslash (D^3)^\perp$ be the Jacobi regularity locus of D , i.e. the set of $\lambda \in W_D$ such that the germ of $J_\gamma(\lambda)$ at λ is convex, where γ is the abnormal extremal passing through λ.

For any $\lambda \in \mathcal{R}_D$, let $\mathcal{W}_{2i}^\lambda$ be the $2i$th Wilczynski invariants of the Jacobi curve J_γ at λ, $1 \leq i \leq n-4$. $\mathcal{W}_{2i}^\lambda$ is a degree $2(i+1)$ homogeneous function on the tangent line to γ at λ.
To any (local) basis (X_1, X_2) of D we assign the following real-valued function on $\mathcal{R}_D$: $A_i^{X_1,X_2}(\lambda) := \mathcal{W}_{2i}^\lambda(\vec{h}_{X_1,X_2}(\lambda))$.
Change of the local baisis causes the multiplication on a positive function depending on the base M only, i.e. the restriction $A_i^{X_1,X_2}$ to $\mathcal{R}_D(q) := \mathcal{R}_D \cap \pi^{-1}(q)$ is the well defined function, up to the multiplication on a positive constant, the ith generalized Wilczynski of D and denoted by A_i.

$A_i|_{\mathcal{R}_D(q)}$ is a degree $2(i+1)$ homogeneous rational function on $(D^2)^\perp(q)$.

Summary on the canonical section of a parametrized curve in $\mathbb{P}V$

Assume that the curve J in a k dimensional projective space $\mathbb{P}V$ of a vector space V is parametrized somehow: $t \mapsto J(t)$.
Let $t \mapsto E(t)$ be a section of J (considered as the tautological bundle over itself).
The convexity assumption is that $E(t), E'(t), \dots, E^{(k)}(t)$ constitute a basis of V.
Among all sections of J there is the unique section, up to a multiplication by a constant, such that

$$\frac{d^{k+1}}{dt^{k+1}}E(t) = \sum_{i=0}^{k-1} B_i(t)\frac{d^i}{dt^i}E(t),$$

called the canonical section of J (i.e. $B_k \equiv 0$) w.r.t. to the chosen parametrization.
If J is self-dual, $k = 2m-1$ and ω is the associated (conformal) symplectic form on V, then $E(t)$ is canonical iff $\omega(E^{(m)}, E^{(m-1)})$ is independent of t.

How to calculate the Wilczynski invariants?

First, on the level of a curve J in Lagrangian Grassmannian $LG(V)$ of a $2m$-dim. symplectic space (V, ω) (keep in mind that for $(2, n)$-distributions $m = n - 3$):

Step 1 Find the osculating flag, in particular, check whether $J^{(1-m)}$ is one-dimensional.

Step 2 Choose some section $\widetilde{E}$ of $J^{(1-m)}$ and a parametrization (not necessarily projective) of J (by a parameter t). "Normalize" it to make it canonical w.r.t. to the chosen parameter. In fact, if $\alpha = \left|\omega\left(\widetilde{E}^{(m)}, \widetilde{E}^{(m-1)}\right)\right|$, then $E := \alpha^{-1/2}\widetilde{E}$ is a canonical section w.r.t. t.

Step 3 Instead of going further with Lagerre-Forsyth normalization to find a projective parameter, use universal polynomial formulas for the densities of Wilczynski invariants in terms of polynomials in the coefficients $\{B_i\}_{i=0}^{2m-2}$ and their derivatives w.r.t. this originally chosen parameter.

Some formula for the first two nontrivial Wilczinski invariants of self-dual curves

If $\dfrac{d^{2m}}{dt^{2m}}E(t) = \sum_{i=0}^{2m-2} B_i(t)\frac{d^i}{dt^i}E(t)$, and $W_{2i}(t) = A_i(t)dt^{2(i+1)}$, then
$A_1 = (2m-2)!\left(\frac{1}{(2m-2)(2m-3)}B_{2m-4} + \frac{(10m+7)}{20(4m^2-1)m}B_{2m-2}^2 - \frac{3}{20}B_{2m-2}''\right)$.

In particular:

- $m = 2$ (the case of $(2,5)$ distributions)
$$A_1 = B_0 + \frac{9}{100}(B_2)^2 - \frac{3}{10}B_2''$$
- $m = 3$ (the case of $(2,6)$ distributions)
$$A_1 = 2\left(B_2 + \frac{37}{175}(B_4)^2 - \frac{9}{5}B_4''\right)$$

If $m = 3$
$A_2 = B_0 + \frac{1}{441}B_2B_4 + \frac{178}{15435}(B_4)^3 - \frac{5}{18}B_2'' - \frac{5}{441}(B_4')^2 - \frac{59}{441}B_4B_4'' + \frac{37}{7}B_4^{(4)}$

On the level of a $(2, n)$-distribution D: Choose a local basis (X_1, X_2) of D again and let $\vec{h} = u_4 \overrightarrow{u}_2 - u_5 \overrightarrow{u}_1$.

$\vec{h}$ defines the parametrization on any abnormal extremal γ and therefore on the Jacobi curve J_γ.

The operation $\frac{d}{dt}$ on sections of J translates to the operation $\operatorname{ad} \vec{h}$ on appropriate vector fields on $\mathbb{P}(D^2)^\perp$ (or $(D^2)^\perp$).

Step 1 For every $\lambda \in (D^2)^\perp$ find the osculating flag of J_γ at λ, in particular, check whether $J^{(4-n)}$ is one-dimensional (in this way you also find the Jacobi regularity set $\mathcal{R}_D$).

Step 2 Choose some section $\widetilde{\mathcal{E}}$ of the line distribution $J^{(4-n)}$. "Normalize" it to make it canonical w.r.t. to the parametrization given by $\vec{h}$: if $\alpha = \left|\sigma\left((\operatorname{ad}\vec{h})^m\widetilde{\mathcal{E}}, (\operatorname{ad}\vec{h})^{m-1}\widetilde{\mathcal{E}}\right)\right|$, then $\mathcal{E} := \alpha^{-1/2}\widetilde{\mathcal{E}}(t)$ is a canonical section $J^{(4-n)}$ w.r.t. the parametrization by $\vec{h}$

Step 3 Find the decomposition of $(\operatorname{ad}\vec{h})^{2m}\mathcal{E}$ in the linear combination w.r.t. $\{(\operatorname{ad}\vec{h})^i\mathcal{E}\}_{i=0}^{2m-2}$ and use the universal polynomial formulas for the density of Wilczynski invariants in terms of universal polynomials in these coefficients and their directional derivative w.r.t. $\vec{h}$.

Suppose the canonical section $\mathcal{E}$ w.r.t. the parametrization by $\vec{h}$ is found, $m := n - 3$, and

$$(\mathrm{ad}\vec{h})^{2m}\mathcal{E} = \sum_{i=0}^{2m-2} \mathcal{B}_i(\mathrm{ad}\,\vec{h})^i\mathcal{E} \mod (\vec{h}, \text{Euler field})\rangle$$

Then the first generalized Wilczynski invariant is given by
$A_1 = (2m-2)!\left(\frac{1}{(2m-2)(2m-3)}\mathcal{B}_{2m-4} + \frac{(10m+7)}{20(4m^2-1)m}\mathcal{B}_{2m-2}^2 - \frac{3}{20}(\mathrm{ad}\,\vec{h})^2(\mathcal{B}_{2m-2})\right)$.
In particular:

- $m = 2$ (the case of $(2,5)$ distributions)
$$A_1 = \mathcal{B}_0 + \frac{9}{100}(\mathcal{B}_2)^2 - \frac{3}{10}(\mathrm{ad}\,\vec{h})^2\mathcal{B}_2(t)$$
- $m = 3$ (the case of $(2,6)$ distributions)
$$A_1 = 2\left(\mathcal{B}_2 + \frac{37}{175}(\mathcal{B}_4)^2 - \frac{9}{5}(\mathrm{ad}\,\vec{h})^2\mathcal{B}_4\right)$$
$A_2 :=$ the last formula on the slide before the last one with $\frac{d}{dt}$ replaced by $\mathrm{ad}\,\vec{h}$.

Discussions on algebraic structure of generalized Wilczynski invariants

For $n = 5$ and s,g,v $(2, 3, 5)$, $\mathbb{P}(D^3)^{\perp} = \emptyset$;
$J^{(4-n)} = \mathcal{V}$, the tangent to the fibers of the bundle $\pi : \mathbb{P}(D^2)^{\perp} \to D$.
If$S_D := W_D \backslash \mathcal{R}_D$, then $S_D = \emptyset$.
The canonical section w,r,t, to $\vec{h}_{X_1, X_2}$ satisfies:
$\mathcal{E}(\lambda) = \gamma_4(\lambda)\partial_{u_4} + \gamma_5(\lambda)\partial_{u_5}$, where $\gamma_4(\lambda)u_5 - \gamma_5(\lambda)u_4 \equiv 1$ (e.g., $\mathcal{E} = \frac{1}{u_5}\partial_{u_4}$ or $-\frac{1}{u_4}\partial_{u_5}$).
The only generalized Wilczynski invariant A_1 is a homogeneous degree 4 polynomial on the fibers and can be computed using the formula in the previous slide.

For $n > 5$ the (complexification of) Jacobi singularity set S_D is not empty and the generalized Wilczynski invariant are not polynomials for generic distributions (they are homogeneous rational functions). This is the case when (complexified) S_D is not empty and the (complexified) characteristic line distribution $\mathcal{C}$ is not tangent to the maximal strata of S_D.

The natural questions are:

- How to describe the singularities/the denominator of the components of the vector field $\mathcal{E}$, which is the canonical section of the curve $J_\gamma^{(4-n)}$ or , equivalently, the set $S_D = W_D \backslash \mathcal{R}_D$, the Jacobi singularity locus or its complexification ?
- How does this set depend on the Tanaka symbol of the distribution?
- What is the algebraic structure of generalized Wilczynski invariant and what more simple invariants can be extracted from it?

Quasi-weights

Assume that D has the small growth vector $(j_1, j_2, \dots j_\mu)$, with $j_1 = 2$, $j_2 = 3$, $j_3 = 5$, $j_\mu = n$, and $j_0 := 0$.
Fix a local frame $(X_1, \dots X_n)$ of TM adapted to the weak derived flag : $D^k = \langle X_1 \dots X_{j_k} \rangle, \quad 1 \le q \le \mu$. Define:

1. the (quasi) impulses $u_i : T^*M \to \mathbb{R}$ of v.f. X_i by $u_i(p, q) = pX_i(q), \quad q \in M, p \in T^*_q M$;
2. the structure functions c_{ij}^k by $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$;
3. the quasi-weights wt for various objects:
   - $\mathrm{wt}(X_i) := k$ if $j_{k-1} < i \le j_k$;
   - $\mathrm{wt}(u_i) = \mathrm{wt}(\vec{u}_i) = -\mathrm{wt}\left(\partial_{u_i}\right) := \mathrm{wt}(X_i)$;
   - $\mathrm{wt}(c_{ij}^k) := \mathrm{wt}(X_i) + \mathrm{wt}(X_j) - \mathrm{wt}(X_k)$; Note that $\mathrm{wt}(c_{ij}^k) \ge 0$ as $[D^i, D^j] \subset D^{i+j}$ and c_{ij}^k depends on the Tanaka symbol iff $\mathrm{wt}(c_{ij}^k) = 0$;
   - The quasi-weights of product of two quasihomogeneous objects is the sum of their quasi-weights (the product might be the usual multiplication or the directional derivative a long a vector field) $\Rightarrow$ quasi-weight of functions and vector fields polynomial/rational w.r.t. the fibers.

Quasihomogeneity of the canonical sections of curves $J_\gamma^{(4-n)}$ induced by Jacobi curves

For example, if $\vec{h} = u_4\vec{u}_2 - u_5\vec{u}_1$ (the generator of the characteristic line distribution $\mathcal{C}$), then $\mathrm{wt}(\vec{h}) = 4$.

Observation 1: *If $\mathcal{E}(\lambda)$ is the canonical section (of the curve $J_\gamma^{(4-n)}(\lambda)$ w.r.t. the parametrization by $\vec{h}$), then E is quasihomogeneous (counting the weight of structures functions) and $\mathrm{wt}(E) = 14 - 4n$, so that the sequence $\{\mathrm{wt}((\mathrm{ad}\vec{h})^j E)\}_{j=0}^{2n-7}$ is an arithmetic progression symmetric w.r.t. to* 0 (exactly as for the spectrum of the action of an elements of the Cartan subalgebra of $\mathfrak{sl}_2$ on a $(2n-6)$-dimensional irreducible $\mathfrak{sl}_2$-module).

- $n = 5$: $\{\mathrm{wt}((\mathrm{ad}\vec{h})^j E)\}_{j=0}^{3} = \{-6, -2, 2, 6\}$;
- $n = 6$: $\{\mathrm{wt}((\mathrm{ad}\vec{h})^j E)\}_{j=0}^{5} = \{-10, -6, -2, 2, 6, 10\}$;
- $n = 7$: $\{\mathrm{wt}((\mathrm{ad}\vec{h})^j E)\}_{j=0}^{7} = \{-14, -10, -6, -2, 2, 6, 10, 14\}$.

The difference in the progression is 4, because $\mathrm{wt}(\vec{h}) = 4$.

Quasihomogeneity of generalized Wilczynski invariants

Observation 2: The ith generalized Wilczynski invariant is a rational function on the fibers of $(D^2)^\perp$ which is quasihomogeneous (counting the structure functions), and $wt(A_i) = 8(i+1)$.

Explanation: $\underbrace{(\mathrm{ad}\vec{h})^{2n-6}\mathcal{E}}_{\mathrm{wt}=4n-10} = \sum_{j=0}^{2n-8} \mathcal{B}_i \underbrace{(\mathrm{ad}\,\vec{h})^{2n-8-j}\mathcal{E}}_{\mathrm{wt}=4n-18-4j} \mod (\vec{h}, \text{Euler field})\rangle \Rightarrow$

$$\mathrm{wt}A_i = \mathrm{wt}(ad\vec{h})^{2n-6}\mathcal{E} - \mathrm{wt}(\mathrm{a}d\vec{h})^{2n-8-2i}\mathcal{E} = 4n - 10 - (4n - 18 - 8i) = 8(i+1)$$

If A_i is not zero for a flat distribution with a given Tanaka symbol, then in general A_i is not quasihomogeneous in u_i's, i.e. without counting the structure functions.
Besides, A_i is always homogeneous in the usual sense w.r.t. u_i of degree $2(i+1)$.

Natural group action on fibers of TM

The Tanaka symbol of D at a point $q \in M$ is the graded Lie algebra $\mathfrak{m}(q) := \bigoplus_{1 \le j \le \mu} D^j(q)/D^{j-1}(q)$ (here we assume that the s.g.v. is constant near q and that μ is the degree of nonholonomy, $D^\mu = TM$).

Let $GL^+(D(q))$ be the subgroup of $GL(D(q))$ preserving the weak derived flag $\{D^j(q)\}_{j=1}^{\mu}$.

Any $A \in GL^+(D(q))$ induces $\mathrm{gr}A \in GL^+(\mathfrak{m}(q))$:

$$\forall x \in D^j(q)/D^{j-1}(q) \quad (\mathrm{gr}A)x := Ax \mod D^{j-1}.$$

Let $G_0(q) \subset GL(\mathfrak{m}(q))$ be the group of automorphisms of the graded Lie algebra $\mathfrak{m}(q)$, $G_0\big(\mathfrak{m}(q)\big) = \mathrm{Aut}(\mathfrak{m}(q))$ ($\mathrm{Lie}\,(G_0(q))$ is the degree zero component of the Tanaka prolongation of $\mathfrak{m}(q)$).

Let $G^+(q) = \{A \in GL^+(D(q)) : \mathrm{grA} \in G_0(q)\}$

Naturally induced group action on the fibers of $(D^2)^\perp$

Natural filtration on T_q^*M (and therefore on the fibers $(D^2)^\perp(q)$ of $(D^2)^\perp$) dual to the weak derived flag:

$$T_q^*M \supset D^\perp(q) \supset (D^2)^\perp(q) \supset (D^3)^\perp(q) \supset \dots \tag{1}$$

The natural action of $G^+(q)$ on T_qM induces the natural action of $G^+(q)$ on T_q^*M preserving the filtration (1) and , in particular, induces the action on $(D^2)^\perp(q)$.
One can use this action (together with the mentioned usual homogeneity and quasi-homogeneity) to deduce more simple invariants (on the base manifold) via some representation theory (and without precise calculations of those Wilczynski invariants).
We still need to have some information on the Jacobi singular set/the denominator of the canonical section $\mathcal{E}$.

Demonstration of the method on (2,6) distributions with s.g.v. (2,3,5,6)

Each fiber of D is endowed with a conformal structure given by

$$B(X,Y) := [X,[Z,Y]] \bmod D^3, \quad X,Y \in D, Z \in D^2/D$$

(note that $\dim D^2/D = 1$).
Implicitly we use here that for a rank 2 distribution with $\dim D^3 = 5$ D and D^3/D^2 are canonically identified (up to a scalar multiplication) by $X \mapsto [Z,X] \mod D^2$, $X \in D, Z \in D^2/D$.
$B(X,Y) = B(Y,X)$ by Jacobi identity.
Explanation:

$$B(X,Y) = [X,[Z,Y]] \bmod D^3 \overset{Jacobi}{=} \underbrace{[[X,Z],Y]}_{[Y,[Z,X]]} + \underbrace{[Z,[X,Y]]}_{0} \bmod D^3 = B(Y,X).$$

The Tanaka symbol of D at q is determined by the signature of B and there are exactly three Tanaka symbols:

1. elliptic (B is sign definite),
2. hyperbolic (B has signature $(1,1)$),
3. parabolic (B is degenerate, of rank 1).

The Jacobi singularity set and the canonical section of $J_\gamma^{(4-n)}$

One can choose a basis $(X_1, \dots, X_6)$ in the Tanaka symbol such that $\mathfrak{g}^{-1} = \langle X_1, X_2 \rangle$, $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$, $X_5 = [X_2, X_3]$ and the only additional possibly nonzero Lie products of vectors $X_1, \dots X_6$ are

$$[X_1, X_4] = X_6,\ [X_2, X_5] = \varepsilon X_6, \quad \varepsilon \in \{-1, 0, 1\} \tag{2}$$

For general $(2,3,5,6)$ distributions the equality in (2) are $\operatorname{mod} D^3$.
Let $Q(\lambda) := B\left(\pi_* \vec{h}(\lambda), \pi_* \vec{h}(\lambda)\right)$. Then in the chosen local frame $Q(\lambda) = \varepsilon u_4^2 + u_5^2$.
The Jacobi singularity set is $S_D = \{\lambda \in W_D : Q(\lambda) = 0\}$, i.e. $\lambda \in S_D$ iff $\mathcal{C}(\lambda)$ is projected to a null line of B.
The canonical section of $J^{(4-n)}$ can be taken as $\mathcal{E} = \dfrac{1}{Q}\partial_{u_6}$.

The generalized Wilczynski invariants are rational and the case of flat distribution with given Tanaka symbol

The generalized Wilczynski invariants $A_i = \dfrac{P_i}{Q^{2(i+1)}}, i = 1, 2$, where P_i is a polynomial.

Quasi-weight and the usual degree analysis:

1. $\mathrm{wt} A_i = 8(i+1)$ and $\mathrm{wt} Q = 6 \Rightarrow$ $\mathrm{wt}(P_i) = \mathrm{wt}(A_i) + 2(i+1)\mathrm{wt}(Q) = 20(i+1)$;
2. If $\deg$ denotes the usual degree, then $\deg A_i = 2(i+1)$ and $\deg Q = 2 \Rightarrow \deg P_i = \deg A_i + 2(i+1) \deg Q = 6(i+1)$.

In general the denominator is not canceled.

However , if the characteristic distribution $\mathcal{C}$ is tangent to the zero level sets of Q , then A_1 and A_2 are polynomials.
In particular, for the flat distributions with given Tanaka symbol

$$A_1 = \varepsilon^2 \frac{74}{175} u_6^4, \quad A_2 = -\frac{178}{15435} \varepsilon u_6^6, \quad \varepsilon \in \{-1, 0, 1\}.$$

For the flat distribution with parabolic Tanaka symbol ($\varepsilon = 0$) $A_1 = A_2 = 0$.

Group action analysis (elliptic case)

Assume that the Tanaka symbol is elliptic (i.e. $\varepsilon = 1$)
The group $G_0 \equiv CO(2)$, G^+ acts on the fiber $(D^2)^\perp(q)$ as the matrix Lie group with the following matrices in the basis $(\partial_{u_4}, \partial_{u_5}, \partial_{u_6})$:

$$\begin{pmatrix} c^3\cos\theta & c^3\sin\theta & 0 \\ -c^3\sin\theta & c^3\cos\theta & 0 \\ \alpha_4 & \alpha_5 & c^4 \end{pmatrix}, \quad c \neq 0.$$

In particular,

$$u_6 \to c^4 u_6 + \alpha_4 u_4 + \alpha_5 u_5. \tag{3}$$

Analyze the first generalized Wilczynski invariant $A_1 = \dfrac{P_1}{Q^4}$.

Assume that $P_1 = \displaystyle\sum_{i=0}^{4} g_i(u_4, u_5) u_6^i$.

(3) yields:

1. $g_4 \to c^{24} g_4$, i.e. g_4 is a well defined, up to a multiplication by a constant, polynomial on $(D^2)^\perp(q)/(D^3)^\perp(q) \cong D(q)$;
2. $g_3 \to c^{28} g_3 - 4(\alpha_4 u_4 + \alpha_5 u_5) c^{24} g_4$

Recall that $\mathrm{wt}(P_1) = 40, \quad \deg P_1 = 12$

$$\Downarrow$$

$$\mathrm{wt}(g_4) = 24, \quad \deg g_4 = 8$$

This, $\mathrm{wt}(u_4) = \mathrm{wt}(u_5) = 3$, and $8 \times 3 = 24$ implies that $\mathrm{wt}(c_{ij}^k) = 0$ for any c_{ij}^k appearing in g_4, i.e. g_4 is the same as in the case of the flat distribution, i.e. , up to a constant $g_4 = \varepsilon^2 Q^4$

Conclusion A_1 (and actually A_2) is not zero for any $(2, 3, 5, 6)$ distribution with the elliptic or hyperbolic Tanaka symbol.

So, g_4 is fixed (and fixes the symbol).

Now, analyze g_i with $0 \leq i \leq 3$:
$\deg g_i = 12 - i$ so g_i can be uniquely represented as
$g_i = \sum_{j=0}^{6-\lceil i/2 \rceil} h_{ij}(u_4, u_5) Q^j$, where $h_{ij}(u_4, u_5)$ are harmonic polynomials, $\Delta h_{ij} = 0$, $\deg h_{ij} = 12 - i - 2j$.

In particular, $g_3 = h_{34}(u_4, u_5) Q^4 + \sum_{j=0}^{3} h_{3j}(u_4, u_5) Q^j$ with $h_{34}(u_4, u_5)$ being linear.

Then by the transformation $u_6 \to c^4 u_6 + \alpha_4 u_4 + \alpha_5 u_5$ (taking into account that $g_3 \to c^{28} g_3 - 4(\alpha_4 u_4 + \alpha_5 u_5) c^{24} \left(\frac{74}{175}\right) Q^4$) , we can make $h_{34} \equiv 0$ (the Ruffini trick or an analog of completion of squares).

This fixes u_6 up to a constant $\Leftrightarrow$ the direction of $X_6 \mod D^2$ (the analog of the Reeb field in contact geometry) $\Leftrightarrow$ reduction of the group G^+ to the group $CO(2)$ represented by the matrix Lie group with the following matrices in the basis $(\partial_{u_4}, \partial_{u_5}, \partial_{u_6})$:

$$\begin{pmatrix} c^3 \cos\theta & c^3 \sin\theta & 0 \\ -c^3 \sin\theta & c^3 \cos\theta & 0 \\ 0 & 0 & c^4 \end{pmatrix}, \quad c \neq 0.$$

After this reduction:

- the tuple of polynomials $\{g_i\}_{i=0}^3$ transforms to the tuple $\{c^{40-4i} g_i\}_{i=0}^3$,
- the tuple of harmonic polynomials $\{h_{ij}\}_{0 \le i \le 3, 0 \le j \le 6 - \lceil i/2 \rceil}$ (where $h_{34} = 0$ and thus can be excluded) transforms to the tuple $\{c^{40-4i-6j} h_{ij}\}_{0 \le i \le 3, 0 \le j \le 6 - \lceil i/2 \rceil}$

$\deg h_{ij} = 12 - i - 2j$, $\mathrm{wt}(h_{ij}) = 40 - 4i - 6j$

$$\Downarrow$$

Since $\mathrm{wt}(u_4) = \mathrm{wt}(u_5) = 3$, the coefficients of h_{ij} as a polynomial in u_4, u_5 have quasi-weights equal to $\mathrm{wt}(h_{ij}) - 3\deg h_{ij} = 4 - i$, i.e. they are polynomial expressions in the structure functions and their derivatives of quasi-weight $4 - i$.

If $z := u_4 + iu_5$, then

$$h_{ij} = r_{ij}\mathrm{Re}\left(z^{12-i-2j}\right) + s_{ij}\mathrm{Im}\left(z^{12-i-2j}\right).$$

for uniquely defined r_{ij} and s_{ij} (which are already the functions on the base manifolds)-the Fourier coefficients of h_{ij} restricted to the unit circle.

Conclusions

At a point $q \in M$ the tuple of numbers

$$\{r_{ij}, s_{ij}\}_{0 \le i \le 3, 0 \le j \le 6 - \lceil i/2 \rceil, (i,j) \neq (3,4)}$$

is defined up to a transformation

$$\{c^{40-4i-6j} r_{ij}, c^{40-4i-6j} s_{ij}\}_{0 \le i \le 3, 0 \le j \le 6 - \lceil i/2 \rceil, (i,j) \neq (3,4)}$$

for $c \neq 0$ and r_{ij}, s_{ij} are polynomial expressions in the structure functions and their derivatives of quasi-weight $4 - i$.

Similar analysis can be done with the second generalized Wilczynski invariant, giving even more invariants due to higher degrees.

There should be a lot of syzygies between these invariants as one expects only $2(6-2) - 6 = 2$ functional invariants for generic $(2, 6)$-distributions.
The case of hyperbolic Tanaka symbol is completely analogous (with the group $CO(1, 1)$ and the d'Alembertian instead of the group $CO(2)$ and the Laplacian).
The parabolic case is different but still treatable.

Comparison with Tanaka theory on a level of Wilczynski invariants

Symplectification (the construction is independent of Tanaka symbol)

Wilczynski invariants: rational functions on the fibers of $(D^2)^\perp$

The natural action of the group $G^+(q)$ on $(D^2)^\perp(q)$ and representation theory

Fix Tanaka symbol and work with distributions with this Tanaka symbol at every point

Tanaka prolongation procedure

The structure of absolute parallelism (the Cartan connection may not exist)

Invariants as functions and tensor on the base manifold

Absolute parallelism, way one: for $n > 5$ without Tanaka theory (B. Doubrov, I. Z., 2006 & 2009)

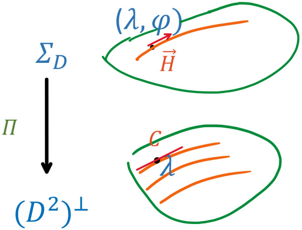
Given $\lambda \in \mathcal{R}_D \subset (D^2)^\perp$ let $\mathfrak{P}_\lambda$ be the set of all projective parametrizations $\varphi : \gamma \mapsto \mathbb{R}$ on the characteristic curve γ, passing through λ, such that $\varphi(\lambda) = 0$.
$\Sigma_D := \{(\lambda, \varphi) : \lambda \in \mathcal{R}_D, \varphi \in \mathfrak{P}_\lambda\}$ is a principal bundle over $\mathcal{R}_D$ with the structural group of $ST(2)$ of all Möbius transformations, preserving 0; $\dim \Sigma_D = 2n - 1$.

Theorem

For any $(2, n)$-distribution, $n > 5$, of maximal class there exists the canonical frame on the corresponding $(2n-1)$-dimensional manifold $\Sigma_D \times \mathbb{Z}_2$. Any $(2, n)$-distribution of maximal class with $(2n-1)$-dimensional Lie algebra of infinitesimal symmetries is locally equivalent to the distribution D_o, associated with the Monge equation $z'(x) = \left(y^{(n-3)}(x)\right)^2$ ($\cong$ parabolic flat for $n = 6$); $\mathrm{symm}(D_o) \cong \mathfrak{gl}_2 \ltimes \mathfrak{n}_{2n-5}$, where $\mathfrak{n}_{2n-5}$ is the $(2n-5)$-dim. Heisenberg algebra.

Explanation: One can construct an Ehresmann connection on the bundle $\Pi : Sigma_D \to \mathcal{R}_D$

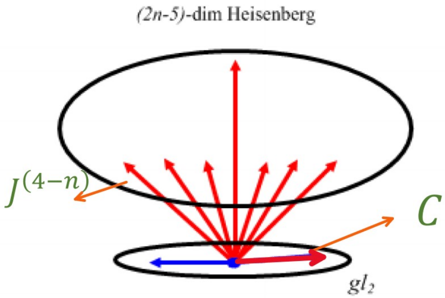
The abnormal extremals are lifted uniquely to Σ_D and the lifts are already parametrized , because the points of the fibers of Σ_D consist of parametrizations, i.e. on Σ_D a well define vector field $\vec{H}$ tangent to the lifts of abnormal extremal is defined.

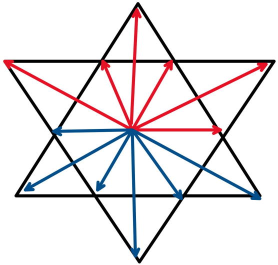


So, for any (λ, φ) consider the canonical section $\mathcal{E}_{\lambda,\varphi}$ of the Jacobi curve through λ w.r.t the parametrization φ, then lift it to Σ_D and produce the moving frame by iterative brackets with $\vec{H}$ and some pairs of them.

Absolute parallelism, way two: for $n \geq 5$ using Tanaka theory but after sympectification

Instead of the original distribution D on M we work with the rank 2 distribution $\mathcal{D} = \mathcal{C} \oplus J^{(4-n)}$ on $\mathbb{P}\mathcal{R}_D \subset \mathbb{P}(D^2)^\perp$.
This distribution have the same Tanaka symbol (see the left picture below) for all $(2, n)$ -distributions of maximal class.
The Tanaka prolongation of this symbol is $\mathfrak{gl}_2 \ltimes \mathfrak{n}_{2n-5}$ if $n > 5$ and G_2 if $n = 5$ (see the right picture with the root diagram for G_2). Tanaka-Morimoto theory for $\mathcal{D} \Rightarrow$ Cartan connection (but over $\mathbb{P}\mathcal{R}_D \subset \mathbb{P}(D^2)^\perp$, not over M).





Absolute parallelism, way three: modified Tanaka theory for flag structures

Instead of the original distribution D on M we work with the contact structure Δ together with a self-dual curve in the projectivization of each fiber of Δ.
Algebraically we start to prolong the algebra

$$\underbrace{\mathrm{symm}(\text{the most symmetric convex curve})}_{\mathfrak{g}_0} \ltimes \underbrace{\text{the Tanaka symbol of}\Delta}_{\mathfrak{g}_-}.$$

The Tanaka symbol of the contact distribution Δ is $\mathfrak{n}_{2n-5}$.

The most symmetric convex curve in projective space $\mathbb{P}\Delta$ is the rational normal curve, represented as $[1 : t : \ldots : t^{2n-7}]$) $\Leftrightarrow$ all Wilczynski invariants of such curve vanish. $\Rightarrow g_0 = \mathfrak{gl}_2$ in an appropriate basis.
The prolongation of $\mathfrak{gl}_2 \ltimes \mathfrak{n}_{2n-5}$ gives the same result as before.

The last two methods can be generalized to distributions of arbitrary rank (B. Doubrov, I. Z., 2016& 2020).

Distributions with vanishing Wilczynski invariants

For the most symmetric rank 2 distributions of maximal class (corresponding to the Monge equation $z'(x) = \left(y^{(n-3)}(x)\right)^2$) all generalized Wilczynski invariants vanish.

Open question: Is the converse true, i.e., from the fact that all generalized Wilzyinski invariants vanish it follows that the distribution is equivalent to the above most symmetric one?

The answer is positive if one restricts to a special class of distribution associated with

$$z'(x) := f(x, y, y', \dots, y^{(n-3)}), \quad F_{y^{(n-3)}y^{(n-3)}} \neq 0 \qquad (4)$$

(B. Doubrov , I.Z. 2011)

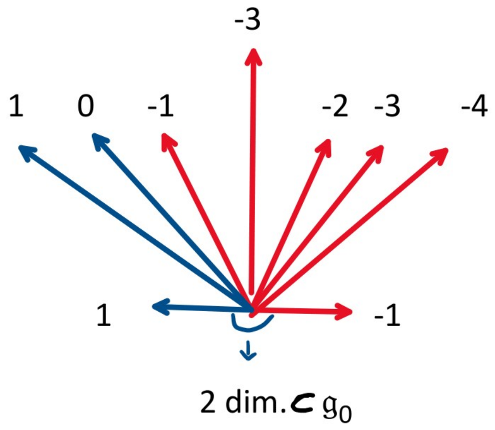
If at least one generalized Wilczynski invariant is non-zero, the maximal infinitesimal symmetry is $(2n-3)$-dimensional and all such distribution are locally classified (B. Doubrov, I. Z., 2014): they correspond to (4) with f being quadratic w.r.t. the derivatives with constant coefficients.

Is the symplectification inevitable/ natural algebraically?

Returning to $(2, 3, 5, 6)$ distribution, if we apply the standard Tanaka theory for distributions with given Tanaka symbol $\mathfrak{m}$:

1. The algebraic prolongation of the ellitpic and hyperbolic symbols are 8-dimensional ($\mathfrak{co}(2) \ltimes m(q)$ and $\mathfrak{co}(1,1) \ltimes m(q)$, respectively); the first algebraic prolongation vanishes; It is expected that in this cases there is a canonical Cartan connection;
2. The flat distribution with parabolic symbol is the most symmetric one with $2 \times 6 - 1 = 11$-dim symmetry algebra$\Rightarrow$ the Tanaka prolongation is 11-dimensional and isomorphic to $\mathfrak{gl}_2 \ltimes \mathfrak{n}_7$ (with an appropriate grading, see pucture on the next slide) :
    - $\dim \mathfrak{g}_0 = 3$: $\mathfrak{g}_0$ is the algebra of triangular 2×2 matrices (G_0 preserves the null line of the (degenerate) canonical quadratic form B).
    - $\dim \mathfrak{g}_1 = 2$.

The grading on the Tanaka prolongation of the parabolic (2,3,5,6) Tanaka symbol



Question: Is there a (linear) normalization condition for the geometric Tanaka prolongation leading to the Cartan connection?

We expect that the answer is NO and then one can try to find a normalization condition having the maximal stabilizer (under the adjoint action of the Lie group corresponding to the nonnegative part of the Tanaka prolongation).

Is this maximal stabilizer equal to T_2 (the group of the triangular nonsingular 2×2 matrices $\Leftrightarrow$ the Borel of GL_2), which is the group corresponding to the nonnegative part of the Tanaka prolongation of the Tanaka symbol of the distribution $\mathcal{D}$ on $\mathbb{P}(D^2)^\perp$ by means of which we constructed the Cartan connection (over $\mathbb{P}(D^2)^\perp$) by the second method?

If yes, it will give an algebraic justification for our symplectification procedure (one requires to make a lift from the original manifold M somewhere ($\mathbb{P}(D^2)^\perp$) to construct the Cartan connection (and therefore invariants) upstairs).

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