Geometry of filtered structures on manifolds: Tanaka's prolongation and beyond

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Question: *When two germs of distributions are equivalent, or, in other words, when two rank ℓ distributions are locally equivalent?*

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The question in this setting: **Given such control system to what more simple system can one transform it by a change of coordinates and a change of a local basis?**

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Therefore, various type of equivalence of differential equations (contact, point etc) can be reformulated as equivalence problems for vector distributions.

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The tuple $(\dim D(q), \dim D^2(q), \dots, \dim D^j(q), \dots)$ is called the *small growth vector of D at the point q* (or, shortly, s.v.g.).

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A distribution D is called *bracket-generating* (*or completely nonholonomic*) if for any $q \in M$ there exist $\mu(q) \in \mathbb{N}$ such that $D^{\mu(q)}(q) = T_qM$. $\Rightarrow$ controllability of the corresponding control system (**Rashevsky-Chow theorem**)

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We expect $\ell(n-\ell) - n$ functional invariants in our equivalence problem.

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Rectification of vector fields	Darboux's normal form $\mathrm{d}x_1-\sum_{i=1}^{[\frac{n-1}{2}]}x_{2i}\mathrm{d}x_{2i+1}=0$	Engel's normal form $\left\{\begin{array}{rcl}\mathrm{d}x_2-x_3\mathrm{d}x_1&=&0\\ \mathrm{d}x_3-x_4\mathrm{d}x_1&=&0\end{array}\right.$

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   $(3, 6)$-distribution with s.g.v. $(3, 6)$ R. Bryant, 1979

Tanaka's approach: main ideas

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Review of Tanaka's theory: the symbol of D at a point

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The universal algebraic prolongation is in fact the kernel of certain coboundary operator for certain Lie algebra cohomology (generalized Spencer or antisymmetrization operator).

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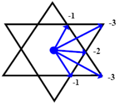

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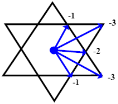

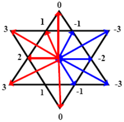

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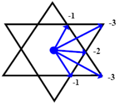

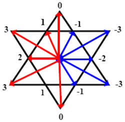

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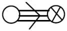

The grading corresponds to the marking of the shorter root in the Dynkin diagram of G_2.

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3. *This upper bound is sharp and is achieved if and only of a distribution is locally equivalent to the flat distribution $D_{\mathfrak{m}}$.*

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Therefore *Tanaka's approach allows one to predict the number of prolongations steps and the dimension of the bundle, where the canonical frame lives,* **without making concrete normalizations on each step** (as the original Cartan method of equivalence suggests)

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2. to generilize the Tanaka prolongation procedure to distributions with nonconstant symbol, because the set of all possible symbols may contain moduli.

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The origin of the method - **Optimal Control Theory**

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Collecting the osculating spaces of this curve of any order together with their skew symmetric complements we assign to the abnormal extremal γ a curve F_γ of isotropic/coisotropic subspaces on the hyperplane $\Delta(\gamma)$ of $T_\gamma N$ called Jacobi curve of the extremal γ.

The role of Jacobi curves

- Any invariant of the Jacobi curve F_γ w.r.t the action of (Conformal) Symplectic Group on the corresponding flag variety of isotropic/coisotropic subspaces (or, shortly, symplectic flags) of $\Delta(\gamma)$ produces an invariant of the distribution D.

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- (compatibility w.r.t. differentiation) $(F_\gamma^i(\lambda)) \subset F_\gamma^{i-1}(\lambda)$

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The tangent vector to the Jacobi curve at a point corresponding to λ can be identified with a line $s_\lambda \subset \mathfrak{csp}\left(\oplus_{\mathrm{i}\in\mathbb{Z}}\mathrm{Gr}^{\mathrm{i}}(\lambda)\right)$ of degree -1, i.e. s.t. $s_\lambda(\mathrm{Gr}^i(\lambda)) \subset \mathrm{Gr}^{i-1}(\lambda))$

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s_λ is called the symbol of the Jacobi curve at λ.

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For fixed rank D and $\dim M$ **the set of all possible symbols of Jacobi curves, up to an isomorphism, is finite**.

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This follows from more general fact (E.Vinberg, 1976): *If* G *is a semisimple Lie group,* $\mathfrak{g}$ *is its Lie algebra with given grading* $\mathfrak{g} = \oplus_{i=-\mu}^{\mu} \mathfrak{g}_i$*, and* G_0 *is the connected subgroup of* G *with the Lie algebra* $\mathfrak{g}_0$*, then the set of orbits of elements of* $\mathfrak{g}_{-1}$ *w.r.t. the adjoint action of* G_0 *is finite.*

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$$\underbrace{s}_{\text{Jacobi symbol of the distribution } D \text{ at } q} \subset \mathfrak{csp}_{-1}(\underbrace{\oplus X^i}_{\text{fixed graded symplectic space } V := \oplus X^i})$$

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2. Jacobi symbols are much coarser characteristic of distributions than Tanaka symbols:
   distributions with different Tanaka symbols and even with different small growth vectors may have the same Jacobi symbol.

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Remark This results can be generalized to natural classes of curves (and submanifolds) in arbitrary parabolic homogeneous spaces G/P (and more general homogeneous spaces if the Lie Algebra of G has fixed grading).

From canonical moving frames for Jacobi curves to canonical frames for distributions

Build the following graded Lie Algebra

$$B(s) = \underbrace{\overbrace{\eta}^{g^{-2}} \oplus \overbrace{\underbrace{(\oplus X^{l})}_{V}}^{g^{-1}}} \oplus \overbrace{\mathfrak{U}_F(s)}^{g^{0}}$$

The Heisenberg algebra -
the Tanaka symbol
of the contact distributionΔ

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Moreover, there exists a distribution with Jacobi symbol s s.t. its algebra of infinitesimal symmetries is isomorphic to $\mathfrak{U}_T(B(s))$ - *symplectically flat distribution with Jacobi symbol s.*

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- The flat curve with symbol s_n^2 is a curve of complete flags consisting of all osculating subspaces of the rational normal curve in $\mathbb{P}^{2n-7}$ $(t \to [1 : t : \ldots : t^{2n-7}])$;
- $\mathfrak{U}_F(s) =$ is the image of the irreducible embedding of $\mathfrak{gl}_2$ into $\mathfrak{gl}_{2n-6}$.

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