Homework assignment 3 — Spring 2016 - MATH622

due March 8, 2016 at the beginning of class

Solve any 5 out of 6 problems below, you can get up to 20 points bonus for solving all problems

1. Problem 10, page 51 of Warner.

2. Problem 18, page 51 of Warner.

3. (*Lagrange multiplier rule*) Let M be a smooth manifold, $g : M \mapsto \mathbb{R}^k$ be a smooth map with components $g_1, g_2, \ldots, g_k$. Further, let c be a regular value of g with $P = g^{-1}(c)$ being non-empty. Let f be a smooth function on M, and suppose that $p \in P$ is a point at which f attains its maximal or minimal value among points in C. Show that there are real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$df_p = \lambda_1 dg_1|_p + \lambda_2 dg_2|_p + \ldots + \lambda_k dg_k|_p.$$

4. Let U be the positive octant of $\mathbb{R}^3$ (i.e. the subset of $\mathbb{R}^3$, where all coordinates are positive). Let D be the distribution on U spanned the vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

   (a) Prove that D is involutive;

   (b) Describe the maximal integral submanifolds of D.

5. Let D be the distribution on $\mathbb{R}^3$ spanned by

$$X = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$

   (a) Find an integral submanifold of D passing through the origin.

   (b) Is D involutive? Explain your answer in light of part (a).

6. Let X_1 and X_2 be two commuting and linearly independent vector fields in a neighborhood of a point m of a d-dimensional manifold M, i.e. $[X_1, X_2] = 0$ in a neighborhood of m and dimension of $\text{span}\Big(X_1(p), X_2(p)\Big)$ is equal to 2 for any p in this neigborhood.

   (a) Prove that there is a coordinate system $(U, x_1, \ldots, x_d)$ around m such that $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = \frac{\partial}{\partial x_2}$ on U.

   (b) What will be a generalization of the statement in part (a) to a larger number of vector fields? Formulate this generalization and give the main ideas of the proof of it.