Homework assignment 5 — Spring 2016 - MATH622

due April 14 2016 at the beginning of class

In the problems 1 and 2 given a smooth map $g : M \mapsto N$ by $g^* : E^*(N) \mapsto E^*(M)$ one defines the pull-back of the differential forms from N to M. In the textbook this map is denoted by δg and it is introduced in subsection 2.22 page 68. We prefer to use notation G^* as more common in the literature.

1. Define a 2-form Ω in $\mathbb{R}^3$ by

$$\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

   (a) Compute Ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

   (b) Compute $d\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

   (c) Compute the restriction $\Omega|_{\mathbb{S}^2} = i^*\Omega$ using coordinates (φ, θ), on the open subset where the coordinates are defined (here $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$ and $i : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is the inclusion map).

   (d) Show that $\Omega|_{\mathbb{S}^2}$ is nowhere zero.

2. Let $g : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$(x, y, z) = g(\theta, \varphi) = \big((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi\big).$$

   Let $\omega = y\, dz \wedge dx$. Compute $g^*(d\,\omega)$ and $d\omega$, and verify by direct computation that $g^*(d\omega) = d(g^*\omega)$.

3. Problem 8, page 78 of Warner.

4. Problem 11, page 78 of Warner (postponed to tne next homework).

5. Problem 16, page 80 of Warner

6. **bonus 20 points** Problem 6, page 78 of Warner.