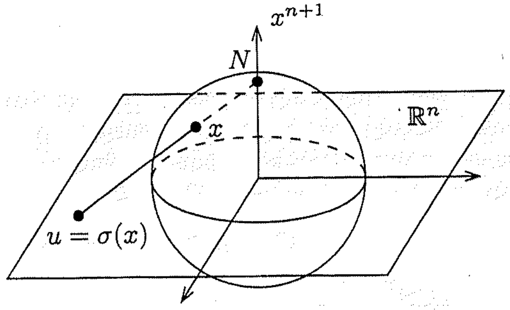

Figure 1.13. Stereographic projection.

Problems

1-1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff. (This space is called the *line with two origins.*)

1-2. Show that the disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second countable.

1-3. Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: Begin by constructing homeomorphisms from $\mathbb{B}^n$ to itself that are smooth on $\mathbb{B}^n \smallsetminus \{0\}$.]

1-4. If k is an integer between 0 and $\min(m, n)$, show that the set of $m \times n$ matrices whose rank is at least k is an open submanifold of $M(m \times n, \mathbb{R})$. Show that this is *not* true if "at least k" is replaced by "equal to k."

1-5. Let $N = (0, \ldots, 0, 1)$ be the "north pole" in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let $S = -N$ be the "south pole." Define *stereographic projection* $\sigma\colon \mathbb{S}^n \smallsetminus \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}.$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \smallsetminus \{S\}$.

(a) For any $x \in \mathbb{S}^n \smallsetminus \{N\}$, show that $\sigma(x)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$, identified with $\mathbb{R}^n$ in the obvious way (Figure 1.13). Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\widetilde{\sigma}$ is called *stereographic projection from the south pole.*)

(b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}.$$

(c) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \smallsetminus \{N\}, \sigma)$ and $(\mathbb{S}^n \smallsetminus \{S\}, \widetilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called *stereographic coordinates.*)

(d) Show that this smooth structure is the same as the one defined in Example 1.20.

1-6. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An *angle function* on a subset $U \subset \mathbb{S}^1$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $e^{i\theta(p)} = p$ for all $p \in U$. Show that there exists an angle function θ on an open subset $U \subset \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

1-7. *Complex projective n-space*, denoted by $\mathbb{CP}^n$, is the set of 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi\colon \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ via $(x^1 + iy^1, \ldots, x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, \ldots, x^{n+1}, y^{n+1})$.)

1-8. Let k and n be integers such that $0 < k < n$, and let $P, Q \subset \mathbb{R}^n$ be the subspaces spanned by $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively, where e_i is the ith standard basis vector. For any k-dimensional subspace $S \subset \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.24 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\binom{I_k}{B}$, where I_k denotes the $k \times k$ identity matrix.

1-9. Let $M = \overline{\mathbb{B}^n}$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary, and that it can be given a natural smooth structure in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point.