INTEGRABLE AFFINE-QUADRATIC SYSTEMS ON LIE GROUPS: GEOMETRY AND MECHANICS

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Abstract

There are two natural "control problems" on semi-simple Lie groups G which admit Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ on their Lie algebras $\mathfrak{g}$ subject to the conditions

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, [\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}, [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}.$$

The first, relevant for the theory of symmetric Riemannian spaces G/K where K is the connectted Lie subgroup of G with its Lie algebra equal to $\mathfrak{k}$, consists of the the sub-Riemannian problem of minimizing the energy $\frac{1}{2}\int_0^T \langle u(t), u(t)\rangle\, dt$ over the horizontal curves $g(t)$ that are the solutions of $\frac{dg}{dt} = g(t)(u(t))$ with $u(t) \in \mathfrak{p}$ where $\langle\, ,\, \rangle$ denotes the left invariant metric on $\mathfrak{p}$ that projects onto the Riemannian metric on G/K.

The second problem utilizes the fact that the Killing form Kl is negative definite on $\mathfrak{k}$ whenever K is compact with finite center. For then $\langle\, ,\, \rangle = -Kl$ is an invariant, non-degenerate quadratic form on $\mathfrak{g}$ that is positive definite on $\mathfrak{k}$ and as such can be used as a base for the following optimal control problem: Minimize $\frac{1}{2}\int_0^T \langle \mathcal{P}(u(t)), u(t)\rangle\, dt$ over the solutions of

$$\frac{dg}{dt} = g(t)(A + u(t)),\; u(t) \in \mathfrak{k}, \tag{1}$$

where A is a fixed element in $\mathfrak{p}$ and where $\mathcal{P}$ is a self-adjoint positive definite operator on $\mathfrak{k}$ relative to $\langle\, ,\, \rangle$.

This problem is called affine-quadratic. It is known that each such problem is well defined for a regular element A in $\mathfrak{p}$, in the sense that for each pair of points g_0 and g_1 there exists a time interval $[0, T]$ and a solution $g(t)$ of (1) that satisfies $g(0) = g_0, g(T) = g_1$ generated by an $L^2([0, T])$ control $u(t)$ which minimizes $\frac{1}{2}\int_0^T \langle \mathcal{P}(u(t)), u(t)\rangle\, dt$ over any other solution of (1) that satisfies the same boundary conditions.

The principal part of the talk is devoted to the Hamiltonian

$$H = \frac{1}{2}\langle \mathcal{P}^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}}\rangle + \langle A, L_{\mathfrak{p}}\rangle, L_{\mathfrak{k}} \in \mathfrak{k}, L_{\mathfrak{p}} \in \mathfrak{p}$$

associated with the above problem. In particular, we will find explicit conditions on A and $\mathcal{P}$ that imply that H is completely integrable in the sense of Liouville. We will

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then interpret our results in the context of contemporary theory of integrable systems on Lie algebras and single out the relevance for some classical problems of mechanics and geometry.