Supplementary lecture notes/ worksheets in MATH433 - Summer 2013 for the classes of Igor Zelenko of June 20 and 21 Definition of determinant using permutations

Print these worksheets and bring them to the classes of June 20 and 21 (we will probably use it already toward the end of June 20 class). Try to read them and to answer the questions below before the class of June 20.

The definition of the determinant of $n \times n$ matrix by induction via the expansion along the first row, given in the standard S. Leon textbook on Linear Algebra (used in all Linear Algebra course at TAMU), page 89 there, is probably the quickest way to introduce this notion. However, this definition is not convenient for explaining the main properties of the determinant and this is probably the main reason why for example Theorem 2.1.1 (in the same page of the Leon book right after this definition) about the expansion along any row and column is neither proved nor even explained there, while the proof of other results (for example the next Theorem 2.1.2) is heavily based on the same unproved Theorem 2.1.1 of the Leon book.

Below I describe another (very standard/common/classical) way to define the determinant. Although this way seems to be more complicated at first glance and needs some additional concepts from Combinatorics, it is significantly elucidate the notion of the determinant and its properties. So, I am convinced that the additional efforts needed to understand this definition and the combinatorial notions related to it will well paid off for the more fundamental understanding of this topic.

1. Inversions and signature of permutation

How are permutations related to the determinant? Let us look more carefully in the expansions of the determinant of 2×2 and 3×3 matrices (that you learned in your previous courses starting from High School Algebra 2 and also in Engineering Calculus 3 and Differential Equations):

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{12} - a_{12}a_{21} \tag{1}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \tag{2}$$

In both cases the determinant is a sum of $n!$ terms of the form

$$\pm a_{1\pi(1)}a_{2\pi(2)} \dots a_{n\pi(n)}$$

($\pm$ means that it is taken either with $+$ or with $-$), where π is a permutation of first n integers.

For example,

- the term $a_{11}a_{22}$ in the right hand side of equation (1) corresponds to the identity permutation $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and it is taken with $+$,
- the term $a_{12}a_{21}$ in the right hand side of equation (1) corresponds to the permutation $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and it is taken with $-$,
- the term $a_{13}a_{21}a_{32}$ in the right hand side of equation (2) corresponds to the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and it is taken with $+$,

Question 3. To what permutation does the term $a_{13}a_{22}a_{31}$ in the right hand side of equation (2) correspond?

Note that it is taken with $-$ there.

The key point is that the same expansion of the determinant is valid for general n. The big question is: **With what sign, $+$ or $-$, the term $a_{1j_1}a_{2j_1}\dots a_{nj_n}$ should be taken?**

In order to answer this question we need the following

Definition 1. *A pair of numbers $(\pi(k), \pi(l))$ in the permutation π is called an inversion if $k < l$ and $\pi(k) > \pi(l)$. In other words, if a larger number precedes a smaller one in the second row of the $2 \times n$ matrix of the permutation then these two numbers create an inversion.*

Definition 2. *The number*

$$\operatorname{sgn}(\pi) = (-1)^{\#\ of\ inversions\ in\ \pi}$$

is called the signature of the permutation π. In other words,

$$\operatorname{sgn}(\pi) = \begin{cases} 1 & \textit{if } \#\textit{of inversions in } \pi \textit{ is even} \\ -1 & \textit{if } \#\textit{of inversions in } \pi \textit{ is odd} \end{cases}$$

In the first case the permutation is called even and in the second case it is called odd.

Remark 1. *The last definition is almost immediately equivalent to the definition on page 165 of Humphreys& Prest (think why?). Our definition is just more elementary and more applicable. The definition in the textbook is more adapted for the subsequent proofs there (for example, of Theorem 4.2.8 there. We will use another strategy and order for proving the same.*

For example,

- In the permutation $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ there is no inversion, so it is an even permutation and its signature is 1;

- In the permutation $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ there is 1 inversion: $(2,1)$, so it is an odd permutation and the signature of it is equal to -1;
- In the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ the inversions are $(3,1)$ and $(3,2)$. So, there are 2 inversions, the permutation is even and its signature is equal to 1;

Question 4. Given the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

- List all its inversions:

- Decide whether it is an even or odd permutation:
- Find its signature:

Question 5. Given the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$

- List all its inversions:

- Decide whether it is an even or odd permutation:
- Find its signature:

Question 6. Given the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}$

- List all its inversions:

- Decide whether it is an even or odd permutation:
- Find its signature:

Now we are ready to give the general definition of the determinant:

Definition 3. *Given an $n \times n$ matrix $A = (a_{ij})$ the determinant $det(A)$ is the following number*

$$\det(A) := \sum_{\pi \in S(n)} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)} \tag{3}$$

In other words, the sign of the term $a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)}$ in the expansion of the determinant is equal to the signature of the permutation π.

Question 7. Show that Definition 3 matches our previous definition of the determinant

- for $n = 2$

- for $n = 3$

I will explain on the whiteboard why the inductive definition via the expansion along the first row given previously matches Definition 3 (take the notes in your notebook).

Question 8. Let $A = (a_{ij})$ be a 5×5 matrix. Determine with what sign appear the term $a_{15}a_{21}a_{32}a_{44}a_{53}$ in the expansion of the determinant of A: For this you need to answer the following questions:

- To what permutation (of first 5 integers) does this term correspond?

- List all inversions of this permutation:

- What is the signature of this permutation and therefore the answer to the original question?:

The expansion (3) is not convenient for calculation of the determinant: you need to perform $n \cdot n!$ multiplications and $n! - 1$ additions/subtractions in order to do that (figure out what are these numbers already for $n = 5$) but it is very convenient in order to understand the properties of the determinant and in this way to justify other, more efficient ways for calculations of the determinant (based on the row/column operations).

2. Signature via transpositions

Recall that a *transposition* is a cycle of length 2. A transposition $(i\, i+1)$, i.e. with adjacent numbers swapped, is called *elementary* or *adjacent transposition.*

Proposition 1. *The product of a permutation π on a transposition τ change the signature of π, i.e* $\operatorname{sgn}(\pi\tau) = -\operatorname{sgn}\pi$.

Proof.

1. *If τ is an elementary transposition then the numbers of inversion of π and $\pi\tau$ differs by* 1 . Indeed, the set $\{\pi\tau(1), \dots \pi\tau(n)\}$ is obtained from the set $\{\pi(1), \dots \pi(n)\}$ by swapping two adjacent elements. If you swap two adjacent elements in $\{\pi(1), \dots \pi(n)\}$ such that the larger one was on the right, then one new inversion is created (and the rest remains the same) and if you swap two adjacent elements such that initially the larger one was on the left, then one inversion disappears (and the rest remains the same). So, in both cases the parity of the permutation changes.

2. Any transposition is a product of an odd number of elementary transpositions. Indeed, if $k < l$ then

$$(k\,l) = \underbrace{(k\,k+1)\dots(l-2\,l-3)(l-1\,l-2)}_{l-k-1\text{times}}\underbrace{(l\,l-1)\dots(k+1\,k+2)(k\,k+1)}_{l-k-1\text{times}},$$

so $(k\,l)$ is the product of $2(l-k)-1$ elementary transpositions. For example for $n \geq 5$

$$(2\,5) = (2\,3)(3\,4)(4\,5)(3\,4)(2\,3)$$

and $(2,5)$ is a product of 5 elementary transpositions. By part 1 of the proof, the multiplication (from the right) by an elementary transposition changes the parity of the permutations. Therefore the product of odd number of elementary transpositions is an odd permutations (note that the identity permutation is even). □

Using Definition 3 and Proposition 1 it is almost immediate to show that **interchanging any row in a matrix changes the sign of the determinant**, then also to prove the part of (unproved) Theorem 2.1.1, page 89 of Leon book regarding the expansion along an arbitrary row and other properties listed in sections 2.1 and 2.2

Note that any cycle is a (nonunique) product of (in general nondisjoint) transpositions. Therefore by cycle decomposition Theorem (Theorem 4.1.3 of our textbook) any permutation π is a (nonunique) product of (in general nondisjoint) transpositions. *Although such decomposition of a permutation in a product of transposition is not unique and, in particular, the number of transposition in different decompositions of the same permutation might be different,* **this number modulo 2 is the same**, *because by the previous proposition the parity of this number is equal to the parity of π, i.e. the parity of the number of inversions in π.*

Using the latter fact one can prove almost immediately the part of (unproved) Theorem 2.1.1, page 89, of the Leon book regarding the expansion along an arbitrary column which in turn shows that all properties of the determinant related to the row operations are automatically valid for the corresponding column operations.

Corollary 1. *The following identity holds:*

$$\mathrm{sgn}(\pi\sigma) = \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma).$$

Hint: Use a decomposition of a permutation into the product of transpositions and Proposition 1 appropriate number of times.