Example 12 from lecture notes 28

Solve: $y''+y=\tan t,\quad 0<t<\frac{\pi}{2} \Rightarrow g(t)=\tan t$

1. Find a fundamental set of solutions of the corresponding homogeneous equation

$y''+y=0 \Rightarrow$ char. equation is $r^2+1=0 \Rightarrow r=\pm i$

$\Rightarrow y_1(t)=\cos t,\quad y_2(t)=\sin t$

$$\Rightarrow \underbrace{\Psi(t)}_{\substack{\text{a fundamental}\\ \text{matrix for the}\\ \text{corresponding system of homogeneous equations}}}=\begin{pmatrix} y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{pmatrix}=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

2. We look for a solution of the original nonhomogeneous equation in the form

$$y(t)=u_1(t)\cos t+u_2(t)\sin t$$

such that

$$\Psi(t)\begin{pmatrix}u_1'(t)\\ u_2'(t)\end{pmatrix}=\begin{pmatrix}0\\ g(t)\end{pmatrix} \quad \text{i.e.}$$

$$\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}u_1'(t)\\ u_2'(t)\end{pmatrix}=\begin{pmatrix}0\\ \tan t\end{pmatrix}. \Rightarrow \text{(using Cramer's rule)}$$

$$u_1'(t)=\frac{\begin{vmatrix}0 & \sin t\\ \tan t & \cos t\end{vmatrix}}{\underbrace{\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}}_{W(\cos t,\sin t)=\cos^2 t+\sin^2 t=1}}=-\frac{\tan t\sin t}{1}=-\frac{\sin^2 t}{\cos t}=-\frac{1-\cos^2 t}{\cos t}=-\sec t+\cos t$$

$$u_2'(t) = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \tan t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = \frac{\cos t \cdot \tan t}{1} = \cancel{\cos t} \cdot \frac{\sin t}{\cancel{\cos t}} = \sin t$$

$$u_1'(t) = -\sec t + \cos t \Rightarrow u_1(t) = \underbrace{-\int \sec t \, dt}_{\text{table integral}} + \int \cos t \, dt$$

table integral
(see below (page 3))
one can find it
<u>If in the test you will have something like this, the antiderivative will be given to you</u>

$$= -\ln(\underbrace{\tan t + \sec t}_{\text{here we use that } 0 \le t < \frac{\pi}{2}}) + \sin t + \underbrace{C_1}_{\text{can be taken } = 0}$$

$$u_2'(t) = \sin t \Rightarrow u_2 = -\cos t + \underbrace{C_2}_{\text{can be taken } = 0}$$

$\Rightarrow$ A particular solution of the original homogeneous equation is

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = (-\ln(\tan t + \sec t) + \sin t)\cos t -$$

$$- \cos t \sin t = -(\ln(\tan t + \sec t))\cos t + \cancel{\sin t \cos t} -$$

$$- \cos t \cancel{\sin t} = -\cos t \ln(\tan t + \sec t) \Rightarrow$$

The general solution of the original homogeneous equation is

$$y(t) = -\cos t \ln(\tan t + \sec t) + \underbrace{C_1 \cos t + C_2 \sin t}_{\text{the general solution of homogeneous equation}}$$

For completeness, the calculation of $\int \sec t \, dt$:

$$\int \sec t \, dt = \int \frac{1}{\cos t} \, dt = \int \frac{du}{\sqrt{1+u^2}}$$

u-substitution: $u = \tan t$

$$du = \frac{1}{\cos^2 t} dt \Rightarrow \frac{1}{\cos t} dt = \cos t \, du$$

$$\frac{1}{\cos^2 t} = 1 + \tan^2 t = 1 + u^2 \Rightarrow \cos t = \frac{1}{\sqrt{1+u^2}} \Rightarrow \frac{1}{\cos t} dt = \frac{1}{\sqrt{1+u^2}} du$$

Now use another change of variables

$$u = \underbrace{\sinh s}_{\text{hyperbolic sine } = \frac{e^s - e^{-s}}{2}} \Rightarrow du = \cosh s \, ds$$

$$1 + u^2 = 1 + \sinh^2 s = \cosh^2 s \Rightarrow \frac{du}{\sqrt{1+u^2}} = \frac{\cancel{\cosh s} \, ds}{\cancel{\cosh s}} = ds$$

$$\Rightarrow \int \frac{du}{\sqrt{1+u^2}} = \int ds = s$$

<u>Returning</u> to variable u: $u = \sinh s = \frac{e^s - e^{-s}}{2} \Rightarrow 2u = \frac{e^{2s} - 1}{e^s} \Rightarrow$

$$e^{2s} - 2ue^s - 1 = 0$$

Let $x = e^s \Rightarrow$ quadratic equation for x, $x > 0$:

$$x^2 - 2ux - 1 = 0 \Rightarrow$$

$$D = 4u^2 + 4 \Rightarrow \underbrace{x}_{\text{positive}} = \frac{2u + \sqrt{4u^2+4}}{2} = u + \sqrt{u^2+1} \Rightarrow$$

$$e^s = u + \sqrt{u^2+1} \Rightarrow s = \ln\left|u + \sqrt{u^2+1}\right|$$

Returning to variable t. $u = \tan t \Rightarrow u^2 + 1 = \tan^2 t + 1 = \frac{1}{\cos^2 t} \Rightarrow$

$\sqrt{u^2+1} = \frac{1}{\cos t} = \sec t \Rightarrow s = \ln|\tan t + \sec t| \Rightarrow \int \sec t = \ln|\tan t + \sec t|$